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On an upwind difference scheme for strongly degenerate parabolic equations modelling the settling of suspensions in centrifuges and non-cylindrical vessels

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Abstract

We prove the convergence of an explicit monotone finite difference scheme approximating an initial-boundary value problem for a spatially one-dimensional quasilinear strongly degenerate parabolic equation, which is supplied with two zero-flux boundary conditions. This problem arises in a model of sedimentation–consolidation processes in centrifuges and vessels with varying cross-sectional area. We formulate the definition of entropy solution of the model in the sense of Kružkov and prove the convergence of the scheme to the unique BV entropy solution of the problem. The scheme and the model are illustrated by numerical examples.

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1. Introduction

The gravity settling of suspensions of flocculated fine particles dispersed in a viscous fluid can be described by an initial-boundary value problem for a scalar strongly degenerate convection–diffusion equation, which arises as a reduced (one-dimensional) case of a multi-dimensional sedimentation–consolidation theory [4,12]. This equation reduces to the scalar conservation law of the well-known kinematic sedimentation model by Kynch [27] when sediment compressibility effects are excluded, as is the case for ideal (non-flocculated) suspensions of small, rigid spherical particles. Since the solutions of strongly degenerate parabolic equations, which include scalar conservation laws as a special case, are discontinuous in general, they need to be defined as weak solutions along with an entropy condition to select the physically relevant weak solution. This property excludes the application of standard numerical schemes for uniformly parabolic equations.

In this paper, we focus on several extensions of the one-dimensional gravity settling model, namely on (1) gravity settling of suspensions in vessels with varying cross-sectional area, (2) centrifugal settling in rotating tubes, and

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(3) centrifugal settling in rotating basket centrifuges. We present a simple upwind numerical finite difference scheme for the simulation of batch settling in these three cases, and prove that for all these cases it converges to the unique entropy solution of the initial-boundary value problem. Since the present paper is a contribution to numerical analysis, let us refer to [9,30] and [1,5,21,22,28,29] for details and various engineering aspects of the model variants (1) and (2, 3), respectively.

To put this contribution in the proper perspective, we mention that a short analysis of the initial-boundary value problem is given in [6] (see also [10]). In [6,10], the existence of BV entropy solutions in the sense of Kružkov [26] and Vol'pert and Hudjaev [33,34] is shown via the vanishing viscosity method, while their uniqueness follows by a technique introduced by Carrillo [15].

On the other hand, Evje and Karlsen [20] show that explicit monotone finite difference schemes, which were first introduced by Crandall and Majda [17] for conservation laws, converge to BV entropy solutions for initial-value problems of strongly degenerate parabolic equations. These results are extended to several space dimensions in [23]. It is the purpose of this contribution to analyze a variant of these techniques suitable for initial-boundary value problems with flux-type boundary conditions and smoothly varying x-dependent coefficients sitting inside the spatial derivatives. An analysis of a similar scheme [7] for a model of continuous sedimentation with slightly different boundary conditions, but with constant cross-sectional area, is presented in [8]. The present paper in part relies on the analyses [8,20,23] but includes new proofs required by our boundary conditions.

Convergence of monotone schemes towards an entropy solution has also been proved for conservation laws and strongly degenerate convection–diffusion equations with discontinuous flux [11,13,14,24,25,31,32]. Such equations arise, for example, if the sedimentation–consolidation model is extended to so-called clarifier–thickener units. From our viewpoint, the convergence proof presented herein is important for the justification of the numerical parameter identification scheme for the sedimentation–consolidation model advanced in [2,3,16].

The remainder of this paper is organized as follows. In Section 2 we state the problem of interest, discuss the significance of the coefficient functions, state the regularity assumptions on them, and recall a known compactness criterion and the use of mollifier functions. In Section 3, we recall the definition of an entropy solution of the initial-boundary value problem. The upwind difference scheme is introduced in Section 4. In Section 5, we derive BV and L^∞ estimates for the numerical solution of the finite difference scheme and the Lipschitz continuity estimates of the discrete integrated diffusion function, while in Section 6, we show that the scheme satisfies a cell entropy inequality which permits to prove that the discrete solutions converge to an entropy solution. Section 7 presents a particular side result of Lemma 5.2 (which is proved in Section 5) stating that monotonically increasing initial data in cylindrical vessels, which are always admitted as a special case herein, lead to discrete solutions (and eventually to entropy solutions) that preserve this property. The settling model and the numerical scheme are illustrated in Section 8. Three technical proofs that arise as straightforward variants of established principles have been deferred to appendices.

2. Statement of the problem and preliminaries

2.1. *The sedimentation–consolidation model*

Let $\widetilde{I} := (a,b)$ denote the range of the spatial coordinate ξ and $\mathcal{T} := (0,T)$ the time interval. Then the one-dimensional sedimentation–consolidation model for centrifuges and vessels with varying cross-sectional area can be stated as the following initial-boundary value problem:

$$\widetilde{S}(\xi)\partial_t\phi + \partial_\xi\big(\widetilde{S}(\xi)\tilde{\nu}(\xi)b(\phi)\big) = \partial_\xi\big(\widetilde{S}(\xi)\partial_\xi A(\phi)\big), \quad (\xi,t)\in\widetilde{Q}_T := \widetilde{I}\times\mathcal{T}, \tag{2.1a}$$

$$\phi(\xi,0) = \tilde{\phi}_0(\xi), \quad \xi\in\widetilde{I}, \tag{2.1b}$$

$$\big(\tilde{\nu}(\xi)b(\phi) - \partial_\xi A(\phi)\big)(\xi_\ell,t) = 0, \quad t\in\mathcal{T},\ \xi_\ell\in\{a,b\}. \tag{2.1c}$$

The sought variable is the volumetric solids concentration ϕ as a function of time t and the spatial coordinate ξ, which either denotes the radial position in a centrifuge or the depth of a settling vessel with varying cross-sectional area. The material specific behaviour of the suspension under study is characterized by the hindered settling func-