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Convergence of a first order scheme for a non-local eikonal equation ☆

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Abstract

We prove the convergence of a first order finite difference scheme approximating a non-local eikonal Hamilton–Jacobi equation. The non-local character of the problem makes the scheme not monotone in general. However, by using in a convenient manner the convergence result for monotone scheme of Crandall–Lions, we obtain the same bound $\sqrt{|\Delta X| + \Delta t}$ for the rate of convergence.

Keywords: Non-local equation; Crandall–Lions error estimate; Eikonal equation; Hamilton–Jacobi equations; Finite difference scheme

1. Introduction

The paper is concerned with the convergence of a first order finite difference scheme that approximates the solution of a non-local Hamilton–Jacobi equation of the form

$$u_t = c[u]|\nabla u| \quad \text{in } \mathbb{R}^2 \times (0, \overline{T}), \qquad u(\cdot,0) = u^0 \quad \text{in } \mathbb{R}^2. \tag{1}$$

The non-local mapping c enjoys suitable regularity assumptions and the initial condition u^0 is globally Lipschitz continuous, possibly unbounded.

A typical example of mapping $c[u]$ we have in mind is

$$c[u] = c^0 \star [u],$$

where $[u]$ is the characteristic function of the set $\{u \geqslant 0\}$, defined by

$$[u] = \begin{cases} 1 & \text{if } u \geqslant 0, \\ 0 & \text{if } u < 0. \end{cases} \tag{2}$$

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Here the kernel c^0, which depends only on the space variables, is integrable and has bounded variation and $\star$ denotes the convolution in space. The zero level set of the solution of the resulting equation

$$\begin{cases} u_t(x,y,t) = (c^0 \star [u](x,y,t))|\nabla u(x,y,t)| & \mathbb{R}^2 \times (0,\overline{T}), \\ u(x,y,0) = u^0(x,y) & \mathbb{R}^2, \end{cases} \tag{3}$$

models the evolution of a dislocation line in a 2D plane (see [2,3] for a physical presentation of the model for dislocation dynamics). Solutions for Eq. (1) should be understood in the framework of the theory of continuous viscosity solutions, see [4,6,7]. Convergence results in the approximation theory of the local version of (1) have been given by Crandall and Lions in [5] for monotone finite difference schemes and by Falcone and Giorgi in [8] and by Falcone and Ferretti in [9] for semi-Lagrangian schemes.

We approximate the non-local equation (1) by a first order finite difference scheme that uses a monotone numerical Hamiltonian for the norm of the spatial gradient, the forward Euler scheme for the time derivative and a proper abstract discrete approximation of the non-local operator c. The non local character of the problem makes the scheme not monotone in general. However, by using in a convenient manner the convergence result for monotone scheme of Crandall Lions, we are able to obtain the same bound $\sqrt{|\Delta X| + \Delta t}$ for the rate of convergence provided the terminal horizon $\overline{T} > 0$ is small enough. The results can be extended to $\mathbb{R}^n$; however, to keep the notation simple and since we have in mind a problem in a plane, we present the problem in $\mathbb{R}^2$.

The present paper is fairly abstract. Its main objective is to find out general assumptions on the discrete approximation of the non-local operator $c[u]$ that guarantee the convergence of the scheme. A companion paper to this article is [1] where we apply this convergence result to the study of the dislocation dynamics equation (3).

The paper is organized as follows. The precise assumptions on the non-local velocity that guarantee the solvability of the non-local Hamilton–Jacobi equation (1) are given in Section 2. In Section 3, we recall the classical finite-difference scheme for the approximation of local eikonal equations. The extension of the scheme to non-local equations is given in Section 5. Section 4 recalls the Crandall–Lions [5] estimate of the rate of convergence for local Hamilton–Jacobi equations. We give an updated proof of the result that tackles the non-classical assumptions we make (in particular, the non-boundedness of the initial condition). Finally, we state and prove in Section 6 our main convergence result.

2. The continuous problem

We are interested in the non-local Hamilton–Jacobi equation

$$u_t = c[u]|\nabla u| \quad \text{in } \mathbb{R}^2 \times (0,\overline{T}), \qquad u(\cdot,0) = u^0 \quad \text{in } \mathbb{R}^2,$$

where the mapping c enjoys suitable regularity assumptions to be specified in a moment and where the initial condition u^0 is (globally) Lipschitz continuous, possibly unbounded.

First, we consider the eikonal equation

$$u_t = c(x,y,t)|\nabla u| \quad \text{in } \mathbb{R}^2 \times (0,\overline{T}), \qquad u(\cdot,0) = u^0 \quad \text{in } \mathbb{R}^2 \tag{4}$$

to set a few notations. We assume that the velocity c is bounded and Lipschitz continuous with respect to all the variables.

The classical theory of viscosity solutions ensures that (4) as a unique continuous viscosity solution with at most linear growth in space. Moreover, the solution u is Lipschitz continuous in space and time, with a Lipschitz constant that depends only on the velocity c and on u^0. We denote by $G : W^{1,\infty}(\mathbb{R}^2 \times [0,\overline{T})) \to \mathrm{Lip}(\mathbb{R}^2 \times [0,\overline{T}))$ the solution operator that associates to c the solution u of (4), i.e.

$$G(c) = u. \tag{5}$$

Here, $\mathrm{Lip}(\mathbb{R}^2 \times [0,\overline{T}))$ denotes the set of the globally Lipschitz functions in space and time, possibly unbounded.

Next, we consider two sets $U \subset \mathrm{Lip}(\mathbb{R}^2 \times [0,\overline{T}))$ and $V \subset W^{1,\infty}(\mathbb{R}^2 \times [0,\overline{T}))$. We assume that V is bounded in $W^{1,\infty}(\mathbb{R}^2 \times [0,\overline{T}))$, i.e. that there is a constant K_0 such that

$$|w|_{W^{1,\infty}(\mathbb{R}^2 \times (0,\overline{T}))} \leqslant K_0, \quad \text{for all } w \in V. \tag{6}$$

For any $0 \leqslant T \leqslant \overline{T}$, we set $U_T = U \cap \mathrm{Lip}(\mathbb{R}^2 \times [0,T))$ and $V_T = V \cap W^{1,\infty}(\mathbb{R}^2 \times [0,T))$, i.e. U_T and V_T are the restrictions to $[0,T)$ of the functions in U and V respectively. We suppose that, for all T, U_T and V_T are closed for