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Third Hankel determinant for bounded turning functions of order alpha

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Abstract

The objective of this paper is to obtain an upper bound to the Third Hankel determinant denoted by $H_3(1)$ for certain subclass of univalent functions, using Toeplitz determinants.
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1. Introduction

Let A denote the class of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. For a univalent function in the class A, it is well known that the nth coefficient is bounded by n. The bounds for the coefficients give information about the geometric properties of these functions. For example, the growth and distortion properties of the normalized univalent function are determined by studying the bound of its second coefficient. The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [1] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.2}$$

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This determinant has been considered by several authors in the literature. For example, Noonan and Thomas [2] studied about the second Hankel determinant of areally mean p-valent functions. Noor [3] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with a bounded boundary. Ehrenborg [4] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [5]. One can easily observe that the Fekete-Szegö functional is $H_2(1)$. Fekete-Szegö then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in S$. Ali [6] found sharp bounds to the first four coefficients and sharp estimate for the Fekete-Szegö functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ when $f^{-1} \in \widetilde{ST}(\alpha)$, the class of strongly starlike functions of order α $(0 < \alpha \le 1)$. Further sharp bounds for the functional $|a_2a_4 - a_3^2|$, the Hankel determinant in the case of $q = 2$ and $n = 2$, known as the second Hankel determinant (functional), given by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2, \tag{1.3}$$

were obtained for various subclasses of univalent and multivalent analytic functions by many authors in the literature. For our discussion in this paper, we consider the Hankel determinant in the case of $q = 3$ and $n = 1$, denoted by $H_3(1)$, given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}. \tag{1.4}$$

For $f \in A$, $a_1 = 1$, so that, we have

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$$

and by applying triangle inequality, we obtain

$$|H_3(1)| \le |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{1.5}$$

Incidentally, all of the functionals on the right hand side of the inequality (1.5) have known (sharp) upper bounds except $|a_2a_3 - a_4|$. The sharp upper bound to the second Hankel functional $H_2(2)$ for the subclass RT of S, consisting of functions whose derivative has a positive real part, studied by Mac Gregor [7] was obtained by Janteng [8]. A well known result is that if $f \in RT$ then $|a_k| \le \frac{2}{k}$, for $k \in \{2, 3, \ldots\}$. Also, if $f \in RT$ then $|a_3 - a_2^2| \le \frac{2}{3}$. Further, for the class RT, the best possible sharp upper bound for the functional $|a_2a_3 - a_4|$ and hence the sharp inequality for $|H_3(1)|$ was obtained by Babalola [9].

Motivated by the result obtained by Babalola [9], we obtain an upper bound to the functional $|a_2a_3 - a_4|$ and hence for $|H_3(1)|$, for the function f given in (1.1), when it belongs to the class $RT(\alpha)$, defined as follows.

Definition 1.1. A function $f(z) \in A$ is said to be in the class $RT(\alpha)$ $(0 \le \alpha < 1)$, consisting of functions whose derivative have a positive real part of order α, if it satisfies the condition

$$Re\ f'(z) > \alpha, \quad \forall\, z \in E. \tag{1.6}$$

Choosing $\alpha = 0$, we obtain $RT(0) = RT$.

Some preliminary Lemmas required for proving our result are as follows:

2. Preliminary results

Let $\mathscr{P}$ denote the class of functions consisting of p, such that

$$p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots = 1 + \sum_{n=1}^{\infty} c_nz^n, \tag{2.1}$$

which are regular in the open unit disc E and satisfy Re$p(z) > 0$ for any $z \in E$. Here $p(z)$ is called the Carathèodory function [10].

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