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Journal of the Nigerian Mathematical Society 34 (2015) 243–248

Journal of the Nigerian Mathematical Society

www.elsevier.com/locate/jnnms

Some extensions and generalizations of Eneström–Kakeya theorem

A.A. Mogbademu[a,*], S. Hans[b], J.A. Adepoju[c]

[a] *Department of Mathematics, University of Lagos, Akoka Lagos, Nigeria*
[b] *Department of Applied Science, ITM University, Gurgaon, India*
[c] *Federal University of Petroleum, Effurum, Delta, Nigeria*

Received 28 July 2014; received in revised form 21 March 2015; accepted 23 March 2015
Available online 1 April 2015

Dedicated to dearest mother Mrs. F.M. Mogbademu on her 70th Birthday

Abstract

In this paper, we put restrictions on the coefficients of a polynomial in order to improve the bounds for their zeros in a specific region. Our results extend and generalise a number of previously well known theorems including Eneström–Kakeya theorem.

Keywords: Polynomials; Zeros; Eneström–Kakeya

1. Introduction

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. One of the fundamental problem of finding out the region which contains all or a prescribed number of zeros of a polynomial was first studied by Gauss [1]. He proved:

Theorem 1.1. *If* $P(z) = z^n + \sum_{j=1}^{n-1} a_j z^j$, *where* a_j *are all real, then* $P(z)$ *has all its zeros in* $|z| \le R$, *where (i)* $R = max(1, 2^{\frac{1}{2}} s)$, *s being the sum of positive* a_j *(ii)* $R = max(n2^{\frac{1}{2}}|a_j|)^j$.

In 1829, Cauchy [2] gave more exact bounds for the moduli of zeros of a polynomial than those given by Gauss [1]. He proved the following result.

Theorem 1.2. *All the zeros of the polynomial* $P(z) = \sum_{j=0}^{n} a_j z^j$ *of degree n lie in the circle* $|z| \le R$, *where R is the root of the equation*

$$|a_0| + |a_1|z + |a_2|z^2 + \cdots + |a_{n-1}|z^{n-1} + +|a_n|z^n = 0.$$

Peer review under responsibility of Nigerian Mathematical Society.
* Corresponding author.
E-mail addresses: amogbademu@unilag.edu.ng (A.A. Mogbademu), sunil.hans82@gmail.com (S. Hans), jadi1011@yahoo.com (J.A. Adepoju).

http://dx.doi.org/10.1016/j.jnnms.2015.03.002

Several generalisations and improvements of this result are available in the literature (see [3,4]). The following elegant results on the location of zeros of a polynomial with restricted coefficients is known as the Eneström–Kakeya theorem [5,6].

Theorem 1.3 (*Eneström–Kakeya*). *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n whose coefficients a_j satisfy*

$$a_n \geq a_{n-1} \cdots \geq a_1 \geq a_0 > 0,$$

then all the zeros of $P(z)$ lie in the closed unit disk $|z| \leq 1$.

Joyal, Labella and Rahman [4] extended Theorem 1.3 to polynomials whose coefficients are monotonic but need not be non-negative as follows:

Theorem 1.4. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that*

$$a_n \geq a_{n-1} \cdots \geq a_1 \geq a_0,$$

then all the zeros of $P(z)$ lie in

$$|z| \leq \frac{a_n + |a_0| - a_0}{|a_n|}.$$

Aziz and Zargar [7] relaxed the conditions of Theorem 1.3 and proved the following generalisation of Theorem 1.4.

Theorem 1.5. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $k \geq 1$,*

$$ka_n \geq a_{n-1} \cdots \geq a_1 \geq a_0,$$

then all the zeros of $P(z)$ lie in

$$|z + k - 1| \leq \frac{ka_n + |a_0| - a_0}{|a_n|}.$$

Govil and Rahman [3] considered polynomials whose coefficients are not necessarily real. Infact, they proved the following generalisation of Theorem 1.3.

Theorem 1.6. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, $j = 0, 1, 2, \ldots, n$ such that*

$$\alpha_n \geq \alpha_{n-1} \cdots \geq \alpha_1 \geq \alpha_0 > 0,$$

where $\alpha_n > 0$, then $P(z)$ has all its zeros in

$$|z| \leq 1 + \frac{2}{\alpha} \sum_{j=0}^{n} |\beta_j|.$$

The following generalizations of Theorems 1.4, 1.5 and 1.6 was proved by Govil and Mc-tume [8].

Theorem 1.7. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, $j = 0, 1, 2, \ldots, n$ such that for some $k \geq 1$,*

$$k\alpha_n \geq \alpha_{n-1} \cdots \geq \alpha_1 \geq \alpha_0,$$

where $\alpha_n > 0$, then $P(z)$ has all its zeros in

$$|z + k - 1| \leq \frac{k\alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|\alpha_n|}.$$

Aziz and Zargar [9] obtained some extensions of Theorem 1.3 by relaxing the hypothesis as follows: