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On common fixed points and multipled fixed points of contractive mappings in metric-type spaces

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Abstract

This research work entails the study of the existence of common fixed points of some Ciric classes of contractive mappings in cone b-metric spaces. The main result obtained unifies, improves and generalizes several results in literature including those of Abbas et al. (2010) and Huang and Xu (2012). Furthermore, as a way of applications, the result is used to discuss common coupled, tripled and multipled fixed points of contractive maps defined on cone b-metric spaces, via product cone b-metric spaces.

Keywords: Contractive maps; Fixed point theorems; Multipled fixed points; Cone b-metric spaces

1. Introduction

The Banach contraction principle proved by Banach [1] in a complete metric space was the starting point of exhaustive research in the fixed point theory. Many contractive conditions under which a map or set of maps have fixed points or common fixed points have been studied in metric spaces (see, for example, [2,3]). Generalized metric spaces have also been considered with the introduction of b-metric spaces [4], cone metric spaces [5] and recently, cone b-metric spaces [6]. Recall that a b-metric defined on a nonempty set X is a symmetric function $d : X \times X \to \mathbb{R}_+$ that satisfies the identity of indiscernibles (or coincidence axiom) and a distorted triangle inequality $d(x, z) \leq K[d(x, y) + d(y, z)]\ \forall x, y, z \in X$, where K is a fixed constant greater or equal to 1.

The results in Abbas et al. [7] and Olaleru and Olaoluwa [8] are a comprehensive generalization of many previous works on contractive mappings in cone metric spaces [9,10]. They established conditions under which four maps tied by a contractive condition have a common fixed point.

Huang and Xu [11] presented some new examples in cone b-metric spaces and proved some fixed point theorems of contractive mappings without the assumption of normality in cone b-metric spaces. In this paper, we generalize the results of Abbas et al. [7] to the context of cone b-metric spaces. Furthermore, the use of functions instead of constants in the contractive conditions studied improves and unifies most results, along this research interest, in literature.

The following definitions and results will be needed in the sequel.

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Definition 1 (*See [5]*)**.** Let E be a real Banach space. A subset P of E is called a cone if and only if:

(a) P is closed, non-empty and $P \neq \{0\}$;
(b) $a, b \in R, a, b \geq 0, x, y \in P$ imply that $ax + by \in P$;
(c) $P \cap (-P) = \{0\}$.

Given a cone P, define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x \ll y$ for $y - x \in int\, P$, where $int\, P$ stands for interior of P. Also we will use $x < y$ to indicate that $x \leq y$ and $x \neq y$.

The cone P in a normed space E is called normal whenever there is a real number $k > 0$, such that for all $x, y \in E, 0 \leq x \leq y$ implies $\|x\| \leq k\|y\|$. The least positive number satisfying this norm inequality is called the normal constant of P.

In the following, we always suppose that E is a Banach space, P is a cone in E with $int(P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 2 (*See [5]*)**.** Let X be a non-empty set and let E be a real Banach space equipped with the partial ordering $\leq$ with respect to the cone $P \subset E$. Suppose that the mapping $d : X \times X \longrightarrow E$ satisfies:

(c_1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and ony if $x = y$;
(c_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(c_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 3 (*See [6]*)**.** Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \to E$ is said to be cone b-metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

(b_1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and ony if $x = y$;
(b_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(b_3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

The pair (X, d) is called a cone b-metric space.

Obviously, cone b-metric spaces generalize b-metric spaces and cone metric spaces. Here are some examples:

Example 4. Let $X = \{1, 2, \ldots, n\}$; $E = \mathbb{R}^2$; $P = \{(x, y) \in E : x \geq 0, y \leq 0\}$. Define $d : X \times X \to E$ by

$$d(x, y) = \begin{cases} \left(\dfrac{1}{|x - y|}, -|x - y|\right) & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

If $n \notin \{2, 3\}$, then (X, d) is a cone b-metric space with the coefficient $s = \frac{(n-1)(n-2)}{2n-3} > 1$ and not a cone metric space since the triangle inequality fails for the points $1, 2, n$. If $n \in \{2, 3\}$, then (X, d) is a cone metric space.

Example 5 (*See [6]*)**.** Let $X = l^p$ with $0 < p < 1$, where $l^p = \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. Let $d : X \times X \to \mathbb{R}_+$ be defined by $d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}}$. Then (X, d) is a b-metric space. Put $E = l^1$, $P = \{\{x_n\} \in E : x_n \geq 0,\ \forall n \geq 1\}$. Letting $\bar{d} : X \times X \to E$ be defined by $\bar{d}(x, y) = \left\{\frac{d(x,y)}{2^n}\right\}_{n \geq 1}$, $(X, \bar{d})$ is a cone b-metric space with the coefficient $s = 2^{\frac{1}{p}} > 1$ but it is not a cone metric space.

Definition 6 (*See [6]*)**.** Let (X, d) be a cone b-metric space, $\{x_n\}$ a sequence in X and $x \in X$. We say that $\{x_n\}$ is

- a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there is some $k \in \mathbb{N}$ such that, for all $n, m \geq k$, $d(x_n, x_m) \ll c$;
- a convergent sequence if for every $c \in E$ with $0 \ll c$, there is some $k \in \mathbb{N}$ such that, for all $n \geq k$, $d(x_n, x) \ll c$. Such x is called limit of the sequence $\{x_n\}$.

Note that every convergent sequence in a cone b-metric space X is a Cauchy sequence. A cone b-metric space X is said to be complete if every Cauchy sequence in X is convergent in X. The following lemma will be needed in the sequel: