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Existence results for a fourth order multipoint boundary value problem at resonance

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Abstract

In this paper we present some existence results for a fourth order multipoint boundary value problem at resonance. Our main tools are based on the coincidence degree theory of Mawhin.

Keywords: Fourth order; Multipoint boundary value problem; Resonance; Coincidence degree

1. Introduction

In this paper, we shall discuss the solvability of the multipoint boundary value problem

$$x^{(iv)}(t) = f(t, x(t), x'(t), x''(t), x'''(t)) \tag{1.1}$$

$$x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i) \qquad x'(0) = x''(0) = 0, \qquad x(1) = x(\eta) \tag{1.2}$$

where $f : [0, 1] \times \mathbb{R}^4 \to \mathbb{R}$ is a continuous function $\alpha_i (1 \leq i \leq m-2) \in \mathbb{R}$, $0 < \xi_1 \leq \xi_2 \leq \cdots < \xi_{m-2} < 1$ and $\eta \in (0, 1)$.

Multipoint boundary value problems of ordinary differential equations arise in a variety of different areas of Applied Mathematics, Physics and Engineering. For example Bridges of small sizes are often designed with two supported points, which leads to a standard two-point boundary condition and bridges of Large sizes are sometimes contrived with multipoint supports which corresponds to a multipoint boundary condition.

Boundary value problem (1.1)–(1.2) is called a problem at resonance if $Lx = x^{(iv)}(t) = 0$ has non-trivial solutions under the boundary conditions (1.2) that is, when $\dim \ker L \geq 1$. On the interval [0, 1] second order and third order boundary value problems at resonance have been studied by many authors (see [1–4]) and references therein.

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Although the existing literature on solutions of multipoint boundary value problems is quite large, to the best of our knowledge there are few papers that have investigated the existence of solutions of fourth order multipoint boundary value problems at resonance. Our motivation for this paper is derived from these previous results.

In what follows, we shall use the classical spaces $C^k[0,1], k = 1, 2, 3$. For $x \in C^3[0,1]$ we use the norm $|x|_\infty = \max_{t\in[0,1]} |x(t)|$. We denote the norm in $L^1[0,1]$ by $|\ |_1$ and on $L^2[0,1]$ by $|\ |_2$. We will use the Sobolev spaces $W^{4,1}(0,1)$ which may be defined by

$$W^{4,1}(0,1) = \{x : [0,1] \longrightarrow \mathbb{R} : x, x', x'', x'''\}$$

are absolutely continuous on $[0, 1]$ with $x^{(iv)} \in L^1[0,1]$.

2. Preliminaries

Consider the linear equation

$$Lx = x^{(iv)}(t) = 0 \tag{2.1}$$

$$x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \qquad x'(0) = x''(0) = 0, \qquad x(1) = x(\eta). \tag{2.2}$$

If we consider a solution of the form

$$x(t) = \sum_{i=0}^{3} a_i t^i, \quad a_i \in \mathbb{R}. \tag{2.3}$$

Then this solution exists if and only if

$$a_3(1 - \eta^3) = 0, \quad \eta \in (0, 1). \tag{2.4}$$

In this case (2.1)–(2.2) has non-trivial solutions.

Hence if $Lx = y$ then L is not invertible. Therefore, the problem is said to be at resonance. We shall prove existence results for the boundary value problem (1.1)–(1.2) under the condition (2.4).

We shall apply the continuation Theorem of Mawhin [5] to get our results. We present some preliminaries needed to understand this continuation Theorem.

Let X and Z be real Banach spaces and $L : domL \subset X \longrightarrow Z$ be a linear operator which is Fredholm of index zero and $P : X \longrightarrow X$, $Q : Z \longrightarrow Z$ be continuous projections such that

$$ImP = \ker L, \ker Q = ImL \quad \text{and} \quad X = \ker L \oplus \ker P$$

$Z = ImL \oplus ImQ$. It follows that $L|_{domL\cap\ker P} \longrightarrow ImL$ is invertible and we write the inverse of this map by K_p. Let Ω be an open bounded subset of X such that $domL \cap \Omega \neq \Phi$ and let $N : \bar{\Omega} \longrightarrow Z$ be an L-compact mapping, that is, the maps $QN(\bar{\Omega})$ is bounded and $K_p(I - Q)N : \bar{\Omega} \longrightarrow X$ is compact. In order to obtain our existence results we shall use the following fixed point Theorem of Mawhin.

Theorem 2.1 (*See [5]*)**.** *Let L be a Fredholm operator of index zero and let N be L-compact on $\bar{\Omega}$. Assume that the following conditions are satisfied*

(i) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in [(domL\backslash\ker L) \cap \partial\Omega \times (0, 1)]$
(ii) $Nx \notin ImL$ *for every* $x \in \ker L \cap \partial\Omega$
(iii) $\deg(JQN|_{\ker L\cap\partial\Omega}; \Omega \cap \ker L, 0) \neq 0$ *where* $Q : Z \to Z$ *is a continuous projection as above and* $J : ImQ \to \ker L$ *is an isomorphism. Then the equation* $Lx = Nx$ *has at least one solution in* $domL \cap \bar{\Omega}$.

We shall prove existence results for the boundary value problem (1.1)-(1.2) when

$$\sum_{i=1}^{m-2} \alpha_i \xi_i^3 = 0 \quad \text{and} \quad \sum_{i=1}^{m-2} \alpha_i = 1.$$

Let $X = C^3[0,1]$, $Z = L^1[0,1]$. Let $L : domL \subset X \longrightarrow Z$ be defined by

$$Lx = x^{(iv)}$$