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Finding a continuous-time Markov chain via sparse stochastic matrices in manpower systems

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Abstract

We consider a manpower system with finite discrete non-overlapping states where recruitment is done to replace wastage and to achieve the desired growth. The states of the system are defined in terms of the ranks. Data for the evolution of manpower structure in the system may be obtained at any choice of time instants. The empirical stochastic matrix resulting from the evolution of the system at each time instant is sparse. We propose a transition model for the system where the multi-step empirical stochastic matrix is expressed as the exponential of a Markov generator. We give illustrations using academic staff data in a university setting.

MSC: 60J20; 91D35

Keywords: Manpower systems; Markov generator; Sparse matrix; Stochastic matrix

1. Introduction

Consider a manpower system stratified into finite discrete non-overlapping states defined in terms of the ranks. Transitions within the system follow a natural order (i.e., from one state to the next higher state). Let $S = \{1, 2, \ldots, k\}$ be the set of these states. The state of the system at any choice of time instant t_v, $v = 0, 1, 2, \ldots, \gamma$, is represented by the row vector

$$\bar{\mathbf{q}}(t_v) = [\bar{q}_1(t_v), \bar{q}_2(t_v), \ldots, \bar{q}_k(t_v)],$$

where $\bar{q}_i(t_v)$ is the expected relative proportion of members of the system in state i at time t_v. The vector $\bar{\mathbf{q}}(t_v)$ is called the expected relative structure (or simply, the structure) of the system at time t_v. Moreover, when the expected number of members of the system at time t_v, denoted by $\bar{\mathbf{n}}(t_v)$, is known, the expected relative structure is obtained as

$$\bar{\mathbf{q}}(t_v) = \bar{\mathbf{n}}(t_v)[\bar{\mathbf{n}}(t_v)\mathbf{e}']^{-1}, \quad (1)$$

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where $\mathbf{e}'$ is a $k \times 1$ vector of ones. The time instant t_v may not be an integer for $v = 0, 1, 2, \ldots, \gamma$, where γ is the maximum time index for which data are available. Let $\{\mathbf{P}(t_v)\}_{v=0}^{\gamma}$ be the sequence of $k \times k$ empirical transition matrices between the states. Assume a state 0 which denotes the environment outside the manpower system and that recruitment is done to replace wastage and to achieve the desired growth. By these assumptions, we have a Markov population replacement process [1–4]. Let $\{\mathbf{w}(t_v)\}_{v=0}^{\gamma}$ be the sequence of $1 \times k$ vectors of observed loss probabilities and let $\{\mathbf{p}_0(t_v)\}_{v=0}^{\gamma}$ be the sequence of $1 \times k$ vectors of observed recruitment probabilities. The entries in each of the sequences $\{\mathbf{P}(t_v)\}_{v=0}^{\gamma}$, $\{\mathbf{w}(t_v)\}_{v=0}^{\gamma}$ and $\{\mathbf{p}_0(t_v)\}_{v=0}^{\gamma}$ are non-negative. The transition probability entries in each of the sequences $\{\mathbf{P}(t_v)\}_{v=0}^{\gamma}$ and $\{\mathbf{w}(t_v)\}_{v=0}^{\gamma}$ are estimated using the maximum likelihood method [5]. Each observed loss probability vector can be expressed in terms of the empirical transition matrix between the states as

$$\mathbf{w}'(t_v) = (\mathbf{I} - \mathbf{P}(t_v))\,\mathbf{e}', \tag{2}$$

where $\mathbf{I}$ is a $k \times k$ identity matrix.

To extrapolate the structure of the system, we need a transition matrix which is stationary. Note that if the transition process is homogeneous, then the η-period transition matrix can be obtained by raising the one-period transition matrix to η index. Nonetheless, situations may arise where the structure of the system is required at any time instant, e.g., a non-integer period. In this case, raising the one-period transition matrix to a non-integer index may not give a substantive transition matrix, especially when the transition matrix is sparse [6]. This is a problem. In this paper we attempt a solution to the problem by expressing the sequence of sparse empirical stochastic matrices $\{\mathbf{S}(t_v)\}_{v=0}^{\gamma}$ as the exponential of a matrix with row sums equal to zero and non-negative off-diagonal elements within some error distance in the Euclidean sense. This approach is closely related to the well-known embedding problem [6,7]. Stochastic matrices and sparse matrices have gained attention in the literature [8–10]. The embedding problem has been acknowledged to have practical relevance in the modelling of social phenomena [7], educational systems [11] and credit risk behaviour [6,12]. The key theoretical underpinnings of this paper are credited to [11]. The approach is centred on embedding the multi-step empirical stochastic matrix into a stationary continuous-time Markov chain. Several aspects of the continuous-time Markov chain have been studied in the literature. These include: the Markov decision process with discounted cost criterion [13], the stability and rate of convergence [14,15], the quasi-birth–death queues [16], the elapsed time between the chain observations [17] and the similarity class [18].

2. Methodology

Suppose the growth rate of the system is unknown to the researcher, but can be estimated from available data [11]. Let $\hat{g}$ denote the estimated growth rate. Then, by the assumption that recruitment is done to replace wastage and to achieve the desired growth, $\hat{g}$, the total recruits at t_{v+1}, denoted as $R(t_{v+1})$, is expressed as

$$R(t_{v+1}) = \mathbf{n}(t_v)\left((1+\hat{g})\mathbf{I} - \mathbf{P}(t_v)\right)\mathbf{e}'. \tag{3}$$

Using Eqs. (1) and (3), the relative structure of the system in terms of the observed structure $\mathbf{q}(t_v)$ is expressed in a form analogous to [19] as

$$\bar{\mathbf{q}}(t_{v+1}) = \mathbf{q}(t_v)\mathbf{S}(t_v), \quad v = 0, 1, 2, \ldots, \gamma, \tag{4}$$

where $\mathbf{S}(t_v)$ is given as

$$\mathbf{S}(t_v) = \left(\mathbf{P}(t_v) + \left((\mathbf{I} - \mathbf{P}(t_v))\,\mathbf{e}' + \hat{g}\mathbf{e}'\right)\mathbf{p}_0(t_v)\right)(1+\hat{g})^{-1}. \tag{5}$$

The matrix $\mathbf{S}(t_v)$ is stochastic as $\mathbf{S}(t_v)\mathbf{e}' = \mathbf{e}'$. We determine the multi-step empirical stochastic matrix by rewriting Eq. (4) for each $v = 0, 1, 2, \ldots, \gamma$, as [20]:

$$\begin{aligned} \bar{\mathbf{q}}(t_1) &= \mathbf{q}(t_0)\mathbf{S}(t_0) \\ \bar{\mathbf{q}}(t_2) &= \mathbf{q}(t_0)\mathbf{S}(t_0)\mathbf{S}(t_1) \\ &\vdots \\ \bar{\mathbf{q}}(t_{\gamma+1}) &= \mathbf{q}(t_0)\prod_{v=0}^{\gamma}\mathbf{S}(t_v). \end{aligned} \tag{6}$$