Available online at www.sciencedirect.com
ScienceDirect
AKCE International Journal of Graphs and Combinatorics 13 (2016) 103–106
AKCE International Journal of Graphs and Combinatorics
www.elsevier.com/locate/akcej

Note on vertex and total proper connection numbers

Emily Chizmar, Colton Magnant*, Pouria Salehi Nowbandegani

Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA, USA

Received 30 August 2014; accepted 11 March 2016
Available online 25 June 2016

Abstract

This note introduces the vertex proper connection number of a graph and provides a relationship to the chromatic number of minimally connected subgraphs. Also a notion of total proper connection is introduced and a question is asked about a possible relationship between the total proper connection number and the vertex and edge proper connection numbers.

Keywords: Colored connectivity; Proper connection; Vertex proper connection

1. Introduction

All graphs considered in this work are simple, finite and undirected. Unless otherwise noted, by a coloring of a graph, we mean a vertex-coloring, not necessarily proper.

Now well studied, the *(edge) rainbow k-connection number* of a graph is the minimum number of colors c such that the edges of the graph can be colored so that between every pair of vertices, there exist k internally disjoint rainbow edge-colored paths. See [1,2] for surveys of results about the rainbow connection number. Note that the rainbow 1-connection number is related, at least conceptually, to the diameter of the graph.

The *total rainbow k-connection number*, defined in [3], is defined to be the minimum number of colors c such that the edges and vertices of the graph can be colored with c colors so that between every pair of vertices, there exist k internally disjoint rainbow paths where here rainbow means all interior vertices and edges have distinct colors. Note that we cannot require the end-vertices of the paths to also have distinct colors as that would reduce the problem to edge rainbow k-connectivity since every vertex would then be required to have a distinct color.

The *edge proper connection number* $\mathrm{pc}_k(G)$, defined in [4] and further studied in [5], is defined to be the minimum number of colors c such that the edges of the graph G can be colored with c colors such that between each pair of vertices, there exist k internally disjoint, properly edge-colored paths. One feature of edge-proper connection that makes the results extremely complicated is that proper edge-colored paths are not transitive in the sense that if there is a proper path from u to v and a proper path from v to w, there may not be a proper path from u to w. For example, let G be a path on three vertices, uvw and color both edges red.

Peer review under responsibility of Kalasalingam University.
* Corresponding author.
E-mail addresses: ec02591@georgiasouthern.edu (E. Chizmar), cmagnant@georgiasouthern.edu (C. Magnant), ps01477@georgiasouthern.edu (P. Salehi Nowbandegani).

http://dx.doi.org/10.1016/j.akcej.2016.06.003

In this work, we consider a vertex version of the edge proper connection number. For a positive integer k, a colored graph G is called *(vertex) properly k-connected* if, between every pair of vertices, there exist at least k internally disjoint properly colored paths. Note that each path, including end-vertices, must be properly colored. Given a graph G, the *vertex proper k-connection number* of the graph G, denoted $\mathrm{vpc}_k(G)$, is the minimum number of colors needed to produce a properly k-connected coloring of G. For ease of notation, let $\mathrm{vpc}(G) = \mathrm{vpc}_1(G)$.

The function $\mathrm{vpc}_k(G)$ is clearly well defined if and only if $\kappa(G) \geq k$. Also note that $\mathrm{vpc}_k(G) \leq \chi(G)$ for every k-connected graph G. Furthermore, the following fact is immediate.

Fact 1. *For all $k \geq 2$ and every k-connected graph G,* $\mathrm{vpc}_k(G) \geq \mathrm{vpc}_{k-1}(G)$.

A graph G is called *minimally k-connected* if G is k-connected but the removal of any edge from G leaves a graph that is not k-connected. A classical result of Mader [6] (also found in [7]) will immediately give us one of our upper bounds.

Theorem 1 ([6,7]). *A minimally k-connected graph is $k + 1$ colorable and this bound is sharp.*

2. General classification

Our first observation demonstrates the transitivity of the vertex proper connection, a fact that is not true in the case of edge proper connection.

Fact 2. *In a colored graph G, if there is a proper path from u to v and a proper path from v to w, then there is a proper path from u to w.*

Proof. The proof is trivial if the u–v path and the v–w path intersect only at v so suppose the paths intersect elsewhere and let x be the first vertex on the path from u to v that is also on the v–w path. Note that we may have $x = u$. Then the subpath of the u–v path that goes from u to x and the subpath of the v–w path that goes from x to w is a properly colored path and completes the proof. ■

Clearly the addition of edges cannot increase the vertex proper connection number of a graph so the following fact is trivial.

Fact 3. *Given a positive integer k and a k-connected graph G, if H is a spanning k-connected subgraph of G, then* $\mathrm{vpc}_k(G) \leq \mathrm{vpc}_k(H)$.

Our main result solidifies the link between the vpc_k function and the chromatic number of the graph. It turns out that $\mathrm{vpc}_k(G)$ always equals the chromatic number of a particular subgraph of G. Let

$$s\chi_k(G) = \min\{\chi(H) : H \text{ is a } k\text{-connected spanning subgraph of } G\}.$$

Theorem 2 (*Classification*). *Given a k-connected graph G,* $\mathrm{vpc}_k(G) = s\chi_k(G)$.

Proof. Given a k-connected spanning subgraph H of G with chromatic number ℓ, color this subgraph properly with ℓ colors. Then between every pair of vertices in H, there are at least k internally disjoint properly colored paths. Thus, using Fact 3, $\mathrm{vpc}_k(G) \leq \mathrm{vpc}_k(H) = \ell$ so $\mathrm{vpc}_k(G) \leq s\chi_k(G)$.

Now let $\ell = \mathrm{vpc}_k(G)$ and consider an ℓ-coloring of G which is properly k-connected. Let $\mathscr{P}$ be the set of all proper paths between pairs of vertices (k paths for each pair of vertices). Then the subgraph H of G induced on all the edges of $\mathscr{P}$ spans G, is k-connected and has chromatic number at most ℓ. This means $\mathrm{vpc}_k(G) \geq s\chi_k(G)$, completing the proof. ■

3. Consequences of Theorem 2

Theorem 2 shows that every statement about vpc_k is a statement about the chromatic number of a minimally k-connected subgraph. Particularly, if G is minimally k-connected, then $\mathrm{vpc}_k(G) = \chi(G)$. When the graph is bipartite, we get the following easy observation.