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On friendly index sets of the edge-gluing of complete graph and cycles

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Abstract

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A vertex labeling $f : V(G) \to Z_2$ induces an edge labeling $f^+ : E(G) \to Z_2$ defined by $f^+(xy) = f(x)+f(y)$, for each edge $xy \in E(G)$. For $i \in Z_2$, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ and $e_f(i) = |\{e \in E(G) : f^+(e) = i\}|$. We say f is friendly if $|v_f(0) - v_f(1)| \le 1$. We say G is cordial if $|e_f(1) - e_f(0)| \le 1$ for a friendly labeling f. The set $FI(G) = \{|e_f(1) - e_f(0)| : f \text{ is friendly}\}$ is called the friendly index set of G. In this paper, we investigate the friendly index sets of the edge-gluing of a complete graph K_n and n copies of cycles C_3. The cordiality of the graphs is also determined.

Keywords: Vertex labeling; Friendly labeling; Cordiality

1. Introduction

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. A vertex labeling $f : V(G) \to Z_2$ induces an edge labeling $f^+ : E(G) \to A$ defined by $f^+(xy) = f(x) + f(y)$, for each edge $xy \in E(G)$. For $i \in Z_2$, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ and $e_f(i) = |\{e \in E(G) : f^+(e) = i\}|$. A labeling f of a graph G is said to be friendly if $|v_f(1) - v_f(0)| \le 1$. For a friendly labeling f of a graph G, we define the friendly index of G under f as $|e_f(1) - e_f(0)|$. If $|e_f(1) - e_f(0)| \le 1$, we say G is cordial [1]. The set $FI(G) = \{|e_f(1) - e_f(0)| : f \text{ is friendly}\}$ is called the friendly index set of G [2]. For more related results and open problems, see [3–11]. When the context is clear, we will also drop the subscript f.

Note that if 0 or 1 is in $FI(G)$, then G is cordial. Thus, the concept of friendly index sets is a generalization of cordiality. Cairnie and Edwards [12] have determined the computational complexity of cordial labeling and Z_k-cordial labeling to be NP-complete. Thus, in general, it is difficult to determine the friendly index sets of graphs.

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Keeping the above in view, in this paper, we investigate the full friendly index sets of the edge-gluing of a complete graph K_n and n copies of cycle C_3, denoted $G(n,3)(n \geq 3)$. We let the vertices of K_n be v_j $(j = 1, 2, \ldots, n)$ and the 3-cycles are given by $v_i v_{i+1} u_i (1 \leq i \leq n-1)$ and $v_n v_1 u_n$. Consequently, the cordiality of $G(n,3)$ is also determined.

The following results were obtained in [7].

Lemma 1.1. *If G is any graph of q edges, then $FI(G) \subseteq \{0, 2, 4, \ldots, q\}$ if q is even, and $FI(G) \subseteq \{1, 3, 5, \ldots, q\}$ if q is odd.*

Lemma 1.2. *Any vertex labeling (not necessarily friendly) of a cycle must have $e(1)$ equal to an even number.*

2. Main results

Lemma 2.1. *For $n \geq 3$, we have*

$$\max\{FI(G(n,3))\} = \begin{cases} (5n-9)/2 & \text{for } n = 3, 5, 7 \\ (5n-8)/2 & \text{for } n = 4, 6, 8 \\ n(n-5)/2 & \text{for } n \geq 9. \end{cases}$$

Proof. The graph $G(n,3)$ has $2n$ vertices and $\frac{n(n+3)}{2}$ edges. It suffices to consider the maximum and minimum value of $e(0)$ such that the number of 0-and 1-vertices are both n.

Suppose n is even. We consider two cases.

Case (a). $e(0)$ is maximum. Assume that t of the vertices v_j are labeled with 0 and the remaining $(n-t)$ vertices v_j are labeled with 1. We consider four subcases.

Subcase (1). $t = 0$ or n. In this case, $e(0) = n(n-1)/2$, $e(1) = 2n$. So, $|e(0) - e(1)| = |\frac{n(n-5)}{2}|$.

Subcase (2). $t = n/2$. In this case, without loss of generality, $\max\{e(0)\}$ is attained if $f(v_j) = f(u_j) = 0$ for $j = 1, 2, \cdots, n/2$, and all other vertices are labeled with 1. Hence, $\max\{e(0)\} = \frac{n}{2}(\frac{n}{2} - 1) + 2(n-2) + 2$, and $|e(0) - e(1)| = n(\frac{n}{2} - 1) + 4n - 4 - \frac{n(n+3)}{2} = \frac{3n}{2} - 4$.

Subcase (3). $0 < t < \frac{n}{2}$. In this case, without loss of generality, $\max\{e(0)\}$ is attained if $f(v_j) = f(u_k) = 0$ for $j = 2, 3, \ldots, t+1$, and $k = 1, 2, \ldots, n-t$, and all other vertices are labeled with 1. Hence, $\max\{e(0)\} = t(t-1)/2 + (n-t)(n-t-1)/2 + 2(n-1) + 2t + 2$, and $e(0) - e(1) = t(t-1) + (n-t)(n-t-1) + 8t - \frac{n(n+3)}{2} = 2t^2 + (8-2n)t + \frac{n(n-5)}{2}$. Therefore, maximal of $e(0) - e(1) = \frac{3n}{2} - 6$ or $\frac{n(n-9)}{2} + 10$ when $t = \frac{n}{2} - 1$ or $t = 1$, respectively. Hence, $\max|e(0) - e(1)| = \frac{3n}{2} - 6$ when $n = 4, 6, 8$, and $\max|e(0) - e(1)| = \frac{n(n-9)}{2} + 10$ when $n \geq 10$.

Subcase (4). $\frac{n}{2} < t < n$. In this case, without loss of generality, $\max\{e(0)\}$ is attained if $f(v_j) = f(u_k) = 1$ when $j = 1, 2, \ldots, t$, $k = 1, 2, \ldots, n-t$, and all other vertices are labeled with 1. Hence, $\max\{e(0)\} = t(t-1)/2 + (n-t)(n-t-1)/2 + 2(n-1) + 2(n-t-1) + 2$ and $e(0) - e(1) = t(t-1) + (n-t)(n-t-1) + 8(n-t) - \frac{n(n+3)}{2} = 2t^2 - (8+2n)t + \frac{n(n+11)}{2}$. Therefore, maximal of $|e(0) - e(1)| = \frac{n(n-9)}{2} + 10$ or $\frac{3n}{2} - 6$ when $t = n-1$ or $t = n/2+1$, respectively. Hence, $\max|e(0) - e(1)| = \frac{3n}{2} - 6$ for $n = 4, 6, 8$, and $\max|e(0) - e(1)| = \frac{n(n-9)}{2} + 10$ when $n \geq 10$.

Case (b). $e(1)$ is maximum. Assume that t of the vertices v_j are labeled with 0 and the remaining $(n-t)$ vertices of v_j are labeled with 1. We consider two subcases.

Subcase (1). $t = 0$ or n. In this case, $e(0) = n(n-1)/2$, $e(1) = 2n$. So, $|e(0) - e(1)| = |\frac{n(n-5)}{2}|$.

Subcase (2). $0 < t < n$. In this case, without loss of generality, $\max\{e(1)\}$ is attained if $f(v_j) = f(u_k) = 1$ for $j = 1, 2, \cdots, t$, $k = t+1, t+2, \ldots, n$, and all other vertices are labeled with 1. Hence, $e(0) - e(1) = 2t(n-t) + 4n - 4 - \frac{n(n+3)}{2} = -2t^2 + 2nt + 4n - 4 - \frac{n(n+3)}{2}$ with maximal is attained when $t = n/2$. Therefore, maximal of $e(0) - e(1) = \frac{n^2}{2} + 4n - 4 - \frac{n(n+3)}{2} = \frac{5n-8}{2}$.

Suppose n is odd. We also consider two cases.

Case (a). $e(0)$ is maximum. Assume that t of the vertices v_j are labeled with 0 and the remaining $(n-t)$ vertices v_j are labeled with 1. We consider three subcases.

Subcase (1). $t = 0$ or n. In this case, $e(0) = n(n-1)/2$, $e(1) = 2n$. So, $|e(0) - e(1)| = |\frac{n(n-5)}{2}|$.