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Domination criticality in product graphs

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Abstract

A connected dominating set is an important notion and has many applications in routing and management of networks. Graph products have turned out to be a good model of interconnection networks. This motivated us to study the Cartesian product of graphs G with connected domination number, $\gamma_c(G) = 2, 3$ and characterize such graphs. Also, we characterize the $k - \gamma$-vertex (edge) critical graphs and $k - \gamma_c$-vertex (edge) critical graphs for $k = 2, 3$ where γ denotes the domination number of G. We also discuss the vertex criticality in grids.

Keywords: Cartesian product; Connected domination; Grid; Critical graphs

1. Introduction

A connected dominating set is an important notion and has many applications in routing and management of networks. Graph products have turned out to be a good model of interconnection networks, [1]. This motivated us to study the connected dominating set in product graphs.

Let $G = (V, E)$ be a simple connected graph with $|V| = n$ and $|E| = m$. The neighbourhood of a vertex u is the set $N(u)$ consisting of all vertices v which are adjacent to u. The closed neighbourhood of a vertex u is $N[u] = N(u) \cup \{u\}$. A set $S \subseteq V$ of vertices in a graph G is called a dominating set if every $v \in V$ is either an element of S or is adjacent to an element of S. The domination number, $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G. A dominating set of G is a minimum dominating set if $|S| = \gamma(G)$. A dominating set S is connected if the induced subgraph $\langle S \rangle$ is connected. The minimum cardinality of a connected dominating set is the connected domination number, $\gamma_c(G)$. These notions are discussed in detail in [2]. Edge critical graphs are graphs in which the domination number decreases upon the addition of any missing edge while vertex critical graphs are graphs in which the domination number decreases upon the deletion of any vertex. Following the notations in [3,2] we say that G is $k - P$-edge critical if $P(G) = k$ and $P(G + e) < k$ for each edge $e \notin E(G)$ and G is $k - P$-vertex critical if $P(G) = k$ but for each vertex $v \in G$, $P(G - v) < k$ where $P \in \{\gamma, \gamma_c\}$.

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The Cartesian product $G \square H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and two vertices $u_i v_j$, $u_x v_y$ are adjacent if either $u_i = u_x$ and $v_j - v_y \in E(H)$ or $u_i - u_x \in E(G)$ and $v_j = v_y$. Also, $\gamma(G \square H) \leqslant min\{\gamma(G)|H|, \gamma(H)|G|\}$ and $\gamma(G \square H) \geqslant min\{|G|, |H|\}$, [4].

In [5] Y.C. Chen and Y.L Syu studied the minimum connected dominating set (MCDS) of the n-dimensional hypercubes and n-dimensional star graphs. In [6] Sumner et al. characterized $2-\gamma$-edge critical graphs and disconnected $3-\gamma$-edge critical graphs. For $k \geqslant 4$, a characterization of connected $k-\gamma$-edge critical graphs is not known. In [7] Chen et al. gave a characterization of $2-\gamma_c$-edge critical graphs and gave some conditions for graphs to be critical. In [8] Brigham et al. characterized $2-\gamma$-vertex critical graphs. In [9] Flandrin et al. studied some properties of $3-\gamma$-vertex critical graphs. Some properties of $3-\gamma_c$-vertex critical graphs are discussed in [10]. However for $k \geqslant 3$, no characterization of $k-\gamma$-vertex critical graphs and $k-\gamma_c$-vertex critical graphs is known. In [11] Goncalves et al. studied the domination number of grids, the Cartesian product of paths.

In [12] and [13] we have studied the problems of changing diameter by the addition or the deletion of some edges for the Cartesian product of graphs.

We denote the Cartesian product of graphs as $G \cong H_1 \square H_2$ with $V(H_1) = \{u_1, u_2, \ldots, u_{n_1}\}$, $V(H_2) = \{v_1, v_2, \ldots, v_{n_2}\}$ and $V(H_1 \square H_2) = \{u_1v_1, u_1v_2, \ldots, u_{n_1}v_{n_2}\}$. Since, $H_1 \square K_1 \cong H_1$ we assume that $H_1, H_2 \neq K_1$.

In this paper, we characterize graphs $G \cong H_1 \square H_2$ with $\gamma_c(G) = 2, 3$. Also, we characterize the $k-\gamma$-vertex (edge) critical graphs and $k-\gamma_c$-vertex (edge) critical graphs for $k = 2, 3$ where γ and γ_c denote the domination number and the connected domination number of G respectively. We also discuss the vertex criticality in grids.

2. Domination critical graphs

Theorem 2.1. *Let $G \cong H_1 \square H_2$ be a connected graph. Then $\gamma(G) = 2$ if and only if $H_1 = K_2$ and H_2 is either a C_4 or has a universal vertex.*

Proof. If $G \cong K_2 \square C_4$, then a minimum dominating set of G is $D = \{u_1v_1, u_2v_3\}$.

If $G \cong K_2 \square H_2$, where H_2 has a universal vertex v_i, then a minimum dominating set of G is $D = \{u_1v_i, u_2v_i\}$. Hence, $\gamma(G) = 2$ in both the cases.

Conversely suppose that $\gamma(G) = 2$.

Suppose that both H_1 and H_2 are not complete graphs.

Then, $\gamma(G) \geqslant min\{|V(H_1)|, |V(H_2)|\} \geqslant 3$.

Hence, at least one graph (say) H_1 should be complete.

Let $G \cong K_{n_1} \square H_2$.

Suppose that H_1 is a complete graph of order at least three. If H_2 has a universal vertex, then a minimum dominating set of G contains vertices from each layer of G and $\gamma(G) \geqslant min\{n_1, n_2\}$. If H_2 does not have a universal vertex, then $\gamma(H_2) \geqslant 2$ and a minimum dominating set of G contains vertices from each layer of G and $\gamma(G) \geqslant n_1$. Thus, in both the cases $\gamma(G) \geqslant 3$. Hence, $n_1 = 2$. Thus, $G \cong K_2 \square H_2$.

Let $n_2 \geqslant 2$.

Then, $\gamma(G) \leqslant min\{2\gamma(H_2), n_2\gamma(K_2)\} = min\{2\gamma(H_2), n_2\}$ (1).

From (1), we have $\gamma(G) = 2$, only if H_2 has a universal vertex, since $n_2 \geqslant 2$.

Next, we consider the case when $\gamma(H_2) \geqslant 2$.

Let $n_2 \geqslant 5$.

Suppose that H_2 contains a vertex v_i of degree $(n_2 - 2)$ and v_i is not adjacent to v_j. Then, $\gamma(H_2) = 2$. Now, a minimum dominating set of G is $D = \{u_1v_i, u_2v_i, u_1v_j\}$ and $\gamma(G) = 3$. If H_2 contains a vertex of degree at most $(n_2 - 3)$, then $\gamma(H_2) = 2$. Let v_p be a vertex of degree $(n_2 - 3)$ and is not adjacent to v_q and v_r in H_2. Then, in G the vertices u_1v_i and u_2v_i dominates $2n_2 - 4$ vertices and the remaining four vertices u_1v_q, u_1v_r, u_2v_q and u_2v_r cannot be dominated by a single vertex. Hence, in these cases $\gamma(G) \geqslant 3$. Thus, $n_2 \leqslant 4$.

Now, by an exhaustive verification of all graphs with $n_2 \leqslant 4$ it follows that $G \cong K_2 \square C_4$. □

Corollary 2.2. *Let $G \cong H_1 \square H_2$ be a connected graph. Then G is $2-\gamma$-vertex critical if and only if $G = C_4$.*

Proof. In Theorem 2.1 we have characterized the Cartesian product of graphs with $\gamma(G) = 2$. Hence, we need to prove the theorem only for such Gs.