Available online at www.sciencedirect.com
ScienceDirect
AKCE International Journal of Graphs and Combinatorics 12 (2015) 47–53
AKCE International Journal of Graphs and Combinatorics
www.elsevier.com/locate/akcej

New characterizations of proper interval bigraphs

Ashok Kumar Das*, Ritapa Chakraborty

Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata-700019, India

Received 13 May 2013; received in revised form 31 March 2014; accepted 11 December 2014
Available online 26 July 2015

Abstract

A proper interval bigraph is a bigraph where to each vertex we can assign a closed interval such that the intervals can be chosen to be inclusion free and vertices in the opposite partite sets are adjacent when the corresponding intervals intersect. In this paper, we introduce the notion of astral triple of edges and along the lines of characterization of interval graphs via the absence of asteroidal triple of vertices we characterize proper interval bigraphs via the absence of astral triple of edges. We also characterize proper interval bigraphs in terms of dominating pair of vertices as defined by Corneil et al. Tucker characterized proper circular arc graphs in terms of circularly compatible 1's of adjacency matrices. Sen and Sanyal characterized adjacency matrices of proper interval bigraphs in terms of monotone consecutive arrangement. We have shown an interrelation between these two concepts.

Keywords: Astral triple of edges; Monotone consecutive arrangement; Zero partitionable; Circularly compatible 1's; Dominating pair of vertices

1. Introduction

The class of *interval graphs* (the intersection graph of intervals) is one of the most well studied class of graphs. *Proper interval graphs* are the intersection graphs of intervals in which no interval properly contains another in the interval model. *Unit interval graphs* are the intersection graphs of intervals of unit length. Roberts [1] introduced and showed that these two subclasses of interval graphs are equivalent.

Analogously, interval digraphs and interval bigraphs were introduced in [2] and [3] respectively.

A digraph is an *interval digraph* if to each vertex v there corresponds an ordered pair (S_v, T_v) of closed intervals such that there is an edge from vertex u to vertex v if and only if S_u intersects T_v. The sets S_v and T_v are the *source set* and *sink set* for v. An *interval bigraph* is a bipartite graph representable by assigning each vertex v an interval so that vertices in opposite partite sets are adjacent if and only if their intervals intersect. The *biadjacency matrix* of a bigraph is the submatrix of the adjacency matrix consisting of the rows indexed by one partite set and the columns indexed by the other. In this paper we denote the biadjacency matrix of B by $A(B)$.

As observed in [4] and [5], the two concepts of interval digraphs and interval bigraphs are equivalent.

Peer review under responsibility of Kalasalingam University.
* Corresponding author.
E-mail addresses: ashokdas.cu@gmail.com (A.K. Das), ritapa.chakroborty@gmail.com (R. Chakraborty).

http://dx.doi.org/10.1016/j.akcej.2015.06.007
0972-8600/

Suppose D is a digraph with $V(D) = \{v_1, \ldots, v_n\}$. Define $B(D)$ to be the bipartite graph with fixed partition (X, Y) where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ with $x_i y_j$ an edge in $B(D)$ if and only if there is a directed edge from v_i to v_j in D.

Conversely, an interval bigraph becomes an interval digraph by adding isolated vertices to equalize the partite sets and then arbitrarily combining pairs of vertices from the two partite sets; the sets for two paired vertices in the representation of the bigraph become the source set and sink set of the new combined vertex.

The point is that the adjacency matrix of an interval digraph is the bi-adjacency matrix of an interval bigraph and conversely the bi-adjacency matrix of an interval bigraph becomes the adjacency matrix of an interval digraph when rows or columns of 0s are added to make it a square.

Interval digraphs and bigraphs and their relations to other families of graphs have been extensively studied in [6–9,2].

Sen and Sanyal [10] introduced the concept of proper interval digraphs where no source interval properly contains another source interval and no sink interval properly contains another sink interval. Recently Lundgren and Brown [6] generalized this definition as follows. A bigraph $B = (X, Y, E)$ is a *proper interval bigraph* if its vertices can be represented by a family of intervals I_v, $v \in X \bigcup Y$, with the property that no interval properly contains another and, for $x \in X$ and $y \in Y$, x and y are adjacent in B if and only if I_x and I_y intersect.

A unit interval bigraph is an interval bigraph where all the intervals are of same length. Sen and Sanyal [10] showed that the concepts of proper interval bigraphs and unit interval bigraphs are equivalent.

A *zero partition* of a binary matrix (i.e. a 0, 1 matrix) is a coloring of each zero with R or C in such a way that every R has only 0*s* colored R to its right and every C has only 0*s* colored C below it. A matrix that admits a zero partition after suitable row and column permutation is *zero partitionable*. A *circular arc graph* is the intersection graph of a set of circular arcs of a host circle. Circular arc graphs have been extensively studied by Tucker [11,12] (also see [13]). A circular arc graph is a *proper circular arc graph* if the circular arc representation can be chosen to be inclusion free. If the vertices of a graph G can be covered by two cliques, then we say that G is a *two-clique graph*. Two-clique circular arc graphs have arisen as an important subclass of circular arc graphs and have been characterized in several ways by Tucker [14], Trotter and Moore [15] and Spinard [16].

Following theorem characterizes interval bigraph in two ways.

Theorem 1. *For a bipartite graph B the following statements are equivalent:*

(a) *B is an interval bigraph*

(b) *A(B) is zero partitionable* [2]

(c) *$\bar{B}$ is a two clique circular arc graph with a representation in which no two arcs together cover the host circle* [9].

Lekkerkerker and Boland [17] defined *asteroidal triple of vertices* to be an independent set of three vertices such that each pair of vertices is joined by a path that avoids the neighborhood of the third. They also characterized the interval graphs as chordal graphs which are free from asteroidal triple of vertices.

A set of three edges in a bigraph is said to form an *asteroidal triple of edges* (ATE) [18,4] if for any two of them there is a path between two that avoids the neighborhood of the third. A bigraph that does not contain $2K_2$ as an induced subgraph is called a *Ferrers bigraph*, and its biadjacency matrix is called a *Ferrers matrix*. It is easy to see that a binary matrix containing only one zero is a Ferrers matrix. So every binary matrix can be expressed as the intersection of finite number of Ferrers matrices. The minimum number of Ferrers bigraphs whose intersection is a given bigraph B is called the Ferrers dimension of B. In [5] it was shown that the bigraphs of Ferrers dimension 2 are free from asteroidal triple of edges. Since the class of interval bigraphs is properly contained in the class of bigraphs of Ferrers dimension 2 [2], so we can state

Theorem 2. *If a bigraph B has an asteroidal triple of edges, then B is not an interval bigraph.*

As observed in [18] a bigraph may not contain an ATE but not of Ferrers dimension 2. Thus the converse of the above theorem is not true.

2. Preliminary results

Let $B = (X, Y, E)$ be a bigraph where $|X| = n$ and $|Y| = m$. Then the biadjacency matrix *A(B)* of B has a *monotone consecutive arrangement (MCA)* if and only if it has independent row and column permutations such that