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Cartesian product of two symmetric starter vectors of orthogonal double covers

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Abstract

An orthogonal double cover (ODC) of a graph H is a collection $\mathcal{G} = \{G_v : v \in V(H)\}$ of $|V(H)|$ subgraphs of H such that every edge of H is contained in exactly two members of $\mathcal{G}$ and for any two members G_u and G_v in $\mathcal{G}$, $|E(G_u) \cap E(G_v)|$ is 1 if u and v are adjacent in H and it is 0 if u and v are nonadjacent in H.

In this paper, we are concerned with the Cartesian product of symmetric starter vectors of orthogonal double covers of the complete bipartite graphs and using this method to construct ODCs for new graph classes.

Keywords: Graph decomposition; Orthogonal double cover; Symmetric starter

1. Introduction

Let H be any graph and let $\mathcal{G} = \{G_1, G_2, \ldots, G_{|V(H)|}\}$ be a collection of $|V(H)|$ subgraphs of H. $\mathcal{G}$ is a double cover (DC) of H if every edge of H is contained in exactly two members in $\mathcal{G}$. If $G_i \cong G$ for all $i \in \{1, 2, \ldots, |V(H)|\}$, then $\mathcal{G}$ is a DC of H by G. If $\mathcal{G}$ is a DC of H by G, then $|V(H)||E(G)| = 2|E(H)|$.

A DC $\mathcal{G}$ of H is an orthogonal double cover (ODC) of H if there exists a bijective mapping $\varnothing : V(H) \to \mathcal{G}$ such that for every choice of distinct vertices u and v in $V(H)$, $|E(\varnothing(u)) \cap E(\varnothing(v))|$ is 1 if $uv \in E(H)$ and is 0 otherwise. If $G_i \cong G$ for all $i \in \{1, 2, \ldots, |V(H)|\}$, then $\mathcal{G}$ is an ODC of H by G.

An automorphism of an ODC $\mathcal{G} = \{G_1, G_2, \ldots, G_{|_{V(H)}|}\}$ of H is a permutation $\pi : V(H) \to V(H)$ such that $\{\pi(G_1), \pi(G_2), \ldots, \pi(G_{|V(H)|})\} = \mathcal{G}$, where for $i \in \{1, 2, \ldots, |V(H)|\}$, $\pi(G_i)$ is a subgraph of H with $V(\pi(G_i)) = \{\pi(v) : v \in V(G_i)\}$ and $E(\pi(G_i)) = \{\pi(u)\pi(v) : uv \in E(G_i)\}$. An ODC G of H is cyclic (CODC) if the cyclic group of order $|V(H)|$ is a subgroup of the automorphism group of $\mathcal{G}$, the set of all automorphisms of $\mathcal{G}$. Note that in this case H is necessarily a regular graph of degree $|E(G)|$. Moreover, if H is not complete, G must be disconnected.

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Throughout this article we make use of the usual notation: $K_{m,n}$ for the complete bipartite graph with independent sets of sizes m and n, P_n for the path on n vertices, $G \cup H$ for the disjoint union of G and H, and mG for m disjoint copies of G.

This concept was originally defined for the case where H is a complete graph. We refer the reader to the survey [1] for more details. While in principle any regular graph H is worth considering (e.g., the remarkable case of hypercubes has been investigated in [2]), the choice of $H = K_{n,n}$ is quite natural, also in view of a technical motivation: ODCs in such graphs are of help in order to obtain ODCs of K_n (see [3], p. 48).

An algebraic construction of ODCs via "symmetric starters" (see Section 2) has been exploited to get a complete classification of ODCs of $K_{n,n}$ by G for $n \le 9$: a few exceptions apart, all graphs G are found this way (see [3], Table 1). This method has been applied in [4] to detect some infinite classes of graphs G for which there is an ODC of $K_{n,n}$ by G.

In [5], we studied the ODC of Cayley graphs and we proved the following.

(i) All 3-regular Cayley graphs, except K_4, have ODCs by P_4.

(ii) All 3-regular Cayley graphs on Abelian groups, except K_4, have ODCs by $P_3 \cup K_2$.

(iii) All 3-regular Cayley graphs on Abelian groups, except K_4 and the 3-prism (Cartesian product of C_3 and K_2), have ODCs by $3K_2$.

Much of the research on this subject focused with the detection of ODCs with pages isomorphic to a given graph G. For results on ODCs of graphs, see [1,4]. The other terminology not defined here can be found in [6].

2. Symmetric starters

All graphs here are finite, simple and undirected. Let $\Gamma = \{\gamma_0, \ldots, \gamma_{n-1}\}$ be an (additive) Abelian group of order n. The vertices of $K_{n,n}$ will be labeled by the elements of $\Gamma \times \mathbb{Z}_2$. Namely, for $(v, i) \in \Gamma \times \mathbb{Z}_2$ we will write v_i for the corresponding vertex and define $\{w_i, u_j\} \in E(K_{n,n})$ if and only if $i \ne j$, for all $w, u \in \Gamma$ and $i, j \in \mathbb{Z}_2$.

Let G be a spanning subgraph of $K_{n,n}$ and let $a \in \Gamma$. Then the graph G with $E(G+a) = \{(u+a, v+a) : (u, v) \in E(G)\}$ is called the *a-translate* of G. The length of an edge $e = (u, v) \in E(G)$ is defined by $d(e) = v - u$.

G is called a half starter with respect to Γ if $|E(G)| = n$ and the lengths of all edges in G are different, i.e. $\{d(e) : e \in E(G)\} = \Gamma$. The following three results were established in [3].

Theorem 1. *If G is a half starter, then the union of all translates of G forms an edge decomposition of $K_{n,n}$, i.e. $\bigcup_{a\in\Gamma} E(G + a) = E(K_{n,n})$.*

Here, the half starter will be represented by the vector: $v(G) = (v_{\gamma_0}, \ldots, v_{\gamma_{n-1}})$, where $v_{\gamma_i} \in \Gamma$ and $(v_{\gamma_i})_0$ is the unique vertex $((v_{\gamma_i}, 0) \in \Gamma \times \{0\})$ that belongs to the unique edge of length γ_i.

Two half starter vectors $v(G_0)$ and $v(G_1)$ are said to be orthogonal if $\{v_\gamma(G_0) - v_\gamma(G_1) : \gamma \in \Gamma\} = \Gamma$.

Theorem 2. *If two half starters $v(G_0)$ and $v(G_1)$ are orthogonal, then $G = \{G_{a,i} : (a, i) \in \Gamma \times \mathbb{Z}_2\}$ with $G_{a,i} = G_i + a$ is an ODC of $K_{n,n}$.*

To each of the two edge decompositions we may associate bijectively an $n \times n-$ square with entries belonging to Γ by $L_i = L_i(k, l) i = 0, 1; k, l \in \Gamma$ with $L_i(k, l) = m$ if and only if the edge $\{k_0, l_1\} \in E(G_{m,i})$. For the squares, the orthogonality condition reads as $|\{(L_0(k, l), L_1(k, l)) : k, l \in \Gamma\}| = n^2$. For more details see [1,3,7].

The subgraph G_s of $K_{n,n}$ with $E(G_s) = \{\{u_0, v_1\} : \{v_0, u_1\} \in E(G)\}$ is called the symmetric graph of G. Note that if G is a half starter, then G_s is also a half starter.

A half starter G is called a symmetric starter with respect to Γ if $v(G)$ and $v(G_s)$ are orthogonal.

Theorem 3. *Let n be a positive integer and let G be a half starter represented by $v(G) = (v_{\gamma_0}, \ldots, v_{\gamma_{n-1}})$. Then G is symmetric starter if and only if $\{v_\gamma - v_{-\gamma} + \gamma : \gamma \in \Gamma\} = \Gamma$.*

The above results on ODCs of graphs motivate us to consider ODCs of $K_{mn,mn}$ if we have the ODCS of $K_{n,n}$ by G and ODCs of $K_{m,m}$ by H where G, H are symmetric starters. In this paper, we have completely settled the existence problem of ODCs of $K_{mn,mn}$ which is presented in the next section.