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Total restrained reinforcement in graphs

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Abstract

In this paper we initiate the study of total restrained reinforcement in graphs. The total restrained reinforcement number in a graph G with no isolated vertex, is the minimum number of edges that have to be added to G so that the resulting graph has total restrained domination number less than total restrained domination number of G. We obtain sharp bounds, exact values and characterization for the total restrained reinforcement number of a graph.

Keywords: Domination; Total restrained domination; Reinforcement

1. Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n. We denote the *open neighborhood* of a vertex v of G by $N_G(v)$, or just $N(v)$, and its *closed neighborhood* by $N_G[v] = N[v]$. For a vertex set $S \subseteq V(G)$, $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$. Let S be a set of vertices, and let $u \in S$. We define the *private neighbor set of u, with respect to S, to be* $pn(u, S) = N[u] \setminus N[S \setminus \{u\}]$. A set of vertices S in G is a *dominating set*, if $N[S] = V(G)$. The *domination number*, $\gamma(G)$ of G, is the minimum cardinality of a dominating set of G. A set of vertices S in G is a *total dominating set*, if $N(S) = V(G)$. The *total domination number*, $\gamma_t(G)$ of G, is the minimum cardinality of a total dominating set of G. A set $S \subseteq V(G)$ is a *total restrained dominating set*, denoted TRDS, if every vertex is adjacent to a vertex in S and every vertex in $V(G) \setminus S$ is also adjacent to a vertex in $V(G) \setminus S$. The *total restrained domination number* of G, denoted $\gamma_{tr}(G)$, is the minimum cardinality of a total restrained dominating set of G. A TRDS of cardinality $\gamma_{tr}(G)$ is called a $\gamma_{tr}(G)$-set. For references on domination in graphs see [1–6].

If S is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of G induced by S. We recall that a leaf in a graph is a vertex of degree one, and a support vertex is one that is adjacent to a leaf. Let $S(G)$ be the set of all support vertices in a graph G.

Kok and Mynhardt [7] introduced the *reinforcement number* $r(G)$ of a graph G as the minimum number of edges that have to be added to G so that the resulting graph G' satisfies $\gamma(G') < \gamma(G)$. They also defined $r(G) = 0$ if

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$\gamma(G) = 1$. This idea was further considered for some other varieties of domination such as *fractional domination*, *independent domination*, and *total domination*, [8–11].

In this paper we study reinforcement by considering a variation based on total restrained domination. The *total restrained reinforcement number* $r_{tr}(G)$ of a graph G with no isolated vertex, is the minimum number of edges that have to be added to G so that the resulting graph G' satisfies $\gamma_{tr}(G') < \gamma_{tr}(G)$. We also define $r_{tr}(G) = 0$ if $\gamma_{tr}(G) = 2$. We determine $r_{tr}(G)$ for some classes of graphs and obtain several upper bounds.

With K_n we denote the *complete graph* on n vertices, with P_n the *path* on n vertices, and with C_n the *cycle* of length n. For two positive integers m, n, the *complete bipartite graph* $K_{m,n}$ is the graph with partition $V(G) = V_1 \cup V_2$ such that $|V_1| = m$, $|V_2| = n$ and such that $G[V_i]$ has no edge for $i = 1, 2$, and every two vertices belonging to different partite sets are adjacent to each other. Obviously, for any two integers $m, n \geq 2$, we have $r_{tr}(K_n) = r_{tr}(K_{m,n}) = r_{tr}(W_n) = r_{tr}(F_n) = 0$, where W_n is the wheel and F_n is the fan of order $n \geq 4$.

2. Exact values

We begin with the following proposition.

Proposition 2.1. *Let G be a graph of order $n \geq 4$. Then $r_{tr}(G) = 0$ if and only if $\gamma_{tr}(G) = 2$.*

Proof. If $\gamma_{tr}(G) = 2$, then trivially $r_{tr}(G) = 0$. Assume that $\gamma_{tr}(G) > 2$. By adding to G all edges belonging to $E(\overline{G})$, we obtain a complete graph K_n. Since $n \geq 4$, we deduce that $\gamma_{tr}(K_n) = 2$, and the result follows. □

In the following we obtain the total restrained reinforcement number for paths and cycles.

Observation 2.2 (*[1]*).

(1) *For $n \geq 2$,*

$$\gamma_{tr}(P_n) = n - 2\left\lfloor \frac{n-2}{4} \right\rfloor.$$

(2) *For $n \geq 3$,*

$$\gamma_{tr}(C_n) = n - 2\left\lfloor \frac{n}{4} \right\rfloor.$$

Proposition 2.3. *Let $n \geq 2$ be an integer. Then*

$$r_{tr}(P_n) = \begin{cases} 0, & \text{for } n \in \{2, 3\}, \\ 2, & \text{for } n \equiv 2 \pmod 4 \text{ and } n \geq 6, \\ 1, & \text{otherwise.} \end{cases}$$

Proof. The result is obvious for $n \leq 3$. Let now $n \geq 4$ and $G = P_n = v_1 v_2 \ldots v_n$. For $n \equiv 0, 1 \pmod 4$, by Observation 2.2, $\gamma_{tr}(G + v_1 v_n) < \gamma_{tr}(G)$, giving that $r_{tr}(G) = 1$. For $n \equiv 3 \pmod 4$, let $n = 4k + 3$. Then $\{v_{4i+1}, v_{4i+2} : 0 \leq i \leq k - 1\} \cup \{v_n, v_{n-1}\}$ is a TRDS for $G + v_n v_{n-3}$, giving that $r_{tr}(G) = 1$. Thus we assume that $n \equiv 2 \pmod 4$.

There is an integer k such that $n = 4k+2$. By Observation 2.2, $\gamma_{tr}(G) = 2k+2$. We show that $r_{tr}(G) \neq 1$. Suppose to the contrary that $\gamma_{tr}(G+e) < \gamma_{tr}(G)$ for some $e \in V(\overline{G})$. By Observation 2.2, $e \neq v_1 v_n$. Let $e = xy$, $H = G+e$, and let S be a $\gamma_{tr}(H)$-set. Without loss of generality, assume that $|S| = 2k + 1$. Then a component of $H[S]$ has more than two vertices. In order for S to dominate the maximum number of vertices, we may assume that a component G_1 of $H[S]$ has three vertices, and any other component is a path on two vertices. If $\{v_1, v_n\} \cap \{x, y\} = \emptyset$, then $V(G_1)$ dominates at most 7 vertices of H and any other component of $H[S]$ dominates at most 4 vertices of H. But clearly $\{v_1, v_2, v_{n-1}, v_n\} \subseteq S$. Now S dominates at most $7 + 4(\frac{|S|-3}{2}) - 2 = 4k + 1 < n$ vertices of H, a contradiction. Thus we assume that $\{v_1, v_n\} \cap \{x, y\} \neq \emptyset$. If $\{v_1, v_n\} = \{x, y\}$, then $H = C_n$ and by Observation 2.2, $\gamma_{tr}(H) = 2k + 2$, a contradiction. Thus, without loss of generality, we may assume that $v_1 = x$ and $v_n \neq y$. Then $V(G_1)$ dominates at most 6 vertices of H and any other component of $H[S]$ dominates at most 4 vertices of H. But clearly $\{v_{n-1}, v_n\} \subseteq S$. Thus S dominates at most $6 + 4(\frac{|S|-3}{2}) - 1 < n$ vertices of H, a contradiction. Thus $r_{tr}(G) \geq 2$. On the other hand if D is a $\gamma_{tr}(G)$-set, then $D - \{v_1, v_n\}$ is a TRDS for $G + v_1 v_n + v_2 v_{n-1}$. Hence $r_{tr}(G) = 2$. □