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Power domination of the cartesian product of graphs

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Abstract

In this paper, we first give a brief survey on the power domination of the Cartesian product of graphs. Then we conjecture a Vizing-like inequality for the power domination problem, and prove that the inequality holds when at least one of the two graphs is a tree.

Keywords: Power domination; Cartesian product; Vizing's Inequality; Corona

1. Introduction

Electrical power companies are required to continually monitor the state of their system as defined by a set of variables, for example, the voltage magnitude at loads and the machine phase angle at generators [1]. These variables can be monitored by placing phase measurement units (PMUs) at selected locations in the system. Due to the high cost of a PMU, it is desirable to monitor (observe) the entire system using the least number of PMUs.

To model this optimization problem as a power domination problem, we use a graph to represent an electrical network. A vertex denotes a possible location where PMU can be placed, and an edge denotes a current carrying wire. A PMU measures the state variable (voltage and phase angle) for the vertex at which it is placed and its incident edges and their ends. All these vertices and edges are said to be observed by the PMU. We can apply Ohm's law and Kirchoff's current law to deduce the other three observation rules:

1. Any vertex that is incident to an observed edge is observed.
2. Any edge joining two observed vertices is observed.
3. For $k \geq 2$, if a vertex is incident to k edges such that $k - 1$ of these edges are observed, then all k of these edges are observed.

We consider only graphs without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *spanning subgraph* of a graph G is a graph whose vertex set is the same as that of G, but its edge set is a

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subset of that of G. The *neighborhood* $N(v)$ of a vertex v is the set of all vertices adjacent to v. We denote $N[v]$ for the set $N(v) \cup \{v\}$. For a set $S \subseteq V(G)$, we write $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$.

A set $S \subseteq V(G)$ is said to be a *dominating set* if every vertex in $V(G)\backslash S$ has at least one neighbor in S. A dominating set of minimum cardinality is called a *minimum dominating set*. The *domination number* $\gamma(G)$ is the cardinality of a minimum dominating set of G. For the power system monitoring problem, a set S is defined to be a *power dominating set* (PDS) if every vertex and every edge in G are observed by S after applying the observation rules. The *power domination number* $\gamma_p(G)$ is the minimum cardinality of a power dominating set of G. We will call a power dominating set with minimum cardinality a $\gamma_p(G)$-set.

For a subset S of $V(G)$, we denote by $M(S)$ the set of vertices in G that is monitored by S. The following algorithm is an alternative approach to the observation rules.

Algorithm 1.1 ([2]). Let $S \subseteq V(G)$ be the set of vertices where the PMUs are placed.

1. (Domination)
   Set $M(S) \leftarrow S \cup N(S)$.
2. (Propagation)
   As long as there exists $v \in M(S)$ such that $N(v) \cap (V(G) - M(S)) = \{w\}$, set $M(S) \leftarrow M(S) \cup \{w\}$.

In other words, the set $M(S)$ is obtained from S by first putting into $M(S)$ the vertices from the closed neighborhood of S, and then repeatedly add to $M(S)$ vertices w that have a neighbor v in $M(S)$ such that all the other neighbors of v are already in $M(S)$. After no such vertex w exists, the set monitored by S is constructed, and it can be easily shown that S corresponds to the power dominating set obtained using the observation rules.

The following observation states that it is possible to find a PDS without any end-vertices or vertices of degree two in the set.

Observation 1.2 ([3]). *If G is a graph with maximum degree at least* 3*, then G contains a $\gamma_p(G)$-set in which every vertex has degree at least* 3.

In this paper, we first give a brief survey about the existing results on the power domination of the Cartesian product of graphs. We then proceed to make a Vizing-like inequality conjecture for the power domination problem, and show that the conjecture is true if one of the graphs is a tree. Finally, an application of this result is presented and some problems are proposed.

2. The Cartesian product of graphs

The Cartesian product of two graphs G_1 and G_2, denoted by $G = G_1 \square G_2$, has $V(G) = V(G_1) \times V(G_2) = \{(x_1, x_2) \mid x_i \in V(G_i) \text{ for } i = 1, 2\}$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$, or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$.

Let K_n, P_n, W_n and C_n denote, respectively, the complete graph, path, wheel and cycle of order n; $K_{1,n}$ denotes the star with $n + 1$ vertices such that n of them are end-vertices.

Observation 2.3. $\gamma_p(G) = 1$ *if G is one of the following:*
(i) $P_n \square K_m$*, where* $n, m \geq 1$*;*
(ii) $P_n \square K_{1,m}$*, where* $n, m \geq 1$*;*
(iii) $P_n \square W_m$*, where* $n \geq 1$ *and* $m \geq 4$.

To verify the above observation, all we need to do is to select carefully a vertex that observes the entire graph. The vertex to choose for each of the respective graphs is shown in Fig. 1.

Remark. We can generalize Observation 2.3 and say that for $n \geq 1$, $\gamma_p(P_n \square H) = 1$ if the domination number of H is 1, that is $\gamma(H) = 1$.

Observation 2.4. $\gamma_p(G) = 2$ *if G is one of the following:*
(i) $C_n \square K_m$*, where* $n, m \geq 3$*;*
(ii) $C_n \square K_{1,m}$*, where* $n, m \geq 3$*;*
(iii) $C_n \square W_m$*, where* $n \geq 3$ *and* $m \geq 4$.