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Extrema property of the k-ranking of directed paths and cycles

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Abstract

A k-ranking of a directed graph G is a labeling of the vertex set of G with k positive integers such that every directed path connecting two vertices with the same label includes a vertex with a larger label in between. The *rank number of* G is defined to be the smallest k such that G has a k-ranking. We find the largest possible directed graph that can be obtained from a directed path or a directed cycle by attaching new edges to the vertices such that the new graphs have the same rank number as the original graphs. The adjacency matrix of the resulting graph is embedded in the Sierpiński triangle.

We present a connection between the number of edges that can be added to paths and the Stirling numbers of the second kind. These results are generalized to create directed graphs which are unions of directed paths and directed cycles that maintain the rank number of a base graph of a directed path or a directed cycle.

Keywords: k-ranking; Directed path; Directed cycle; Adjacency matrix; Sierpinski triangle

1. Introduction

A vertex coloring of a directed graph is a labeling of its vertices so that no two adjacent vertices receive the same label. In a directed path, edges are oriented in the same direction. A k-ranking of a directed graph is a labeling of the vertex set with k positive integers such that for every directed path connecting two vertices with the same label there is a vertex with a larger label in between. A ranking is *minimal* if the reduction of any label violates the ranking property. The *rank number* $\chi_k(G)$ of a directed graph G is the smallest k such that G has a minimal k-ranking.

It is known that the rank number of a graph G may increase just by adding a new edge, even if the new edge and G share vertices. This raises the question "what is the maximum size of a directed graph that satisfies the property that its rank number is equal to the rank number of its largest directed subpath?" Flórez and Narayan [1,2] found results related to this question; however, the problem is still open. We believe that studying particular cases will lead to a better understanding of the problem and potential solutions.

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In this paper we study the necessary and sufficient conditions for the largest possible directed graph that can be obtained by attaching edges to either a directed path or directed cycle without changing the rank number and the number of vertices. In [1] there is a solution for the undirected case. Here, we analyze cases for which the new directed graph keeps the rank number of the original graph. The maximum number of edges in such graphs is described as well as which edges are present in the graphs.

We build families of directed graphs by adding directed edges (called *admissible*) to a directed path (called the *base*). Those families satisfy the condition that the graphs are maximal graphs with the property that the rank number of each graph equals the rank number of the base directed path. The graphs of the first two families described were constructed recursively by adding all admissible edges (without increasing the number of vertices) to a base directed path. The same idea is extended to directed cycles.

We generalize the concept developed to build the first four families above to other maximal families of graphs preserving the rank number of the base directed path by adding admissible directed paths and directed cycles to a base directed path or directed cycle.

The number of edges and the number of admissible edges of the graphs in the first four families are counted using known numerical sequences. We prove, using the recursive construction, that the maximum number of edges in some of those families of graphs is given by a Stirling number of the second kind.

For those who are interested in computational matters, we provide algorithms, some of which are given in terms of adjacency matrices. We found an interesting connection between one of the adjacency matrices and the Sierpiński triangle. The adjacency matrix of the first graph found in this paper embeds naturally in the Sierpiński sieve triangle.

2. Preliminary concepts

In this section we review some known concepts and results, introduce new definitions, and give a proof for a lemma.

Let $V := \{v_1, v_2, \ldots, v_\eta\}$ be a set of vertices of a directed graph. An edge (arc) with vertices $\{v_i, v_j\} \subseteq V$, $i < j$, with orientation $v_i \to v_j$ is denoted by $\overrightarrow{e}$ or by $\overrightarrow{v_i v_j}$, and the edge with orientation $v_i \leftarrow v_j$ is denoted by $\overleftarrow{e}$ or by $\overleftarrow{v_i v_j}$. A directed path with vertices V is denoted by $\overrightarrow{P_\eta}$ if its edges are of the form $\overrightarrow{e}$. A directed cycle with vertices V is denoted by $\overrightarrow{C_\eta}$ if its edges are of the form $\overrightarrow{e}$.

Let G be a finite directed graph with vertex set $V(G)$. A *k-ranking* of G is a labeling (or coloring) of $V(G)$ with k positive integers such that every directed path that connects two vertices of the same label (color) contains a vertex of a larger label (color). Thus, a labeling function $f : V(G) \longrightarrow \{1, 2, \ldots, k\}$ is a vertex *k-ranking* of G if for all $u, v \in V(G)$ such that $u \neq v$ and $f(u) = f(v)$, then every directed path connecting u and v contains a vertex w such that $f(w) > f(u)$. Like the chromatic number, the *rank number* of a graph G is defined to be the smallest k such that G has a minimal k-ranking; it is denoted by $\chi_k(G)$.

Let H_1 and H_2 be directed graphs with $V(H_1) \subseteq V(H_2)$ and $E(H_1) \cap E(H_2) = \emptyset$. We say that a directed edge $e \in E(H_1)$ is *admissible* for H_2 if $\chi_r(H_2 \cup \{e\}) = \chi_r(H_2)$, and e is *forbidden* for H_2 if $\chi_r(H_2 \cup \{e\}) > \chi_r(H_2)$.

We distinguish two types of admissible edges. A directed edge e is *admissible of type I* for G if e and the edges in the edge set of G have the same direction. A directed edge e is *admissible of type II* if e and the edges in the edge set of G have opposite direction. Note that admissible edges of type II allow for edges with opposite directions between two vertices.

For example, Fig. 1 shows the graph $\overrightarrow{\mathcal{G}}_4 := \overrightarrow{P}_{2^4-1} \cup H(\overrightarrow{\mathcal{G}}_4)$ where $H(\overrightarrow{\mathcal{G}}_4)$ is the graph formed with all admissible edges of type I for $\overrightarrow{P}_{2^4-1}$. In Fig. 2 we show the graph $\overleftarrow{\mathcal{G}}_4 := \overrightarrow{P}_{2^4-1} \cup H(\overleftarrow{\mathcal{G}}_4)$ where $H(\overleftarrow{\mathcal{G}}_4)$ is the graph formed with all admissible edges of type II for $\overrightarrow{P}_{2^4-1}$. Since $\overrightarrow{P}_{2^4-1}$ gives rise to both graphs $\overrightarrow{\mathcal{G}}_4$ and $\overleftarrow{\mathcal{G}}_4$, they have the same set of vertices $V = \{v_1, \ldots, v_{15}\}$, where v_1 is leftmost vertex and v_{15} is the rightmost vertex. The numbers on the graphs represent the labelings. That is,

$$
\begin{array}{lllll}
f(v_1) = 1; & f(v_2) = 2; & f(v_3) = 1; & f(v_4) = 3; & f(v_5) = 1; \\
f(v_6) = 2; & f(v_7) = 1; & f(v_8) = 4; & f(v_9) = 1; & f(v_{10}) = 2; \\
f(v_{11}) = 1; & f(v_{12}) = 3; & f(v_{13}) = 1; & f(v_{14}) = 2; & f(v_{15}) = 1.
\end{array}
$$

The largest label in $\overrightarrow{P}_{2^4-1}$ is 4. So, 4 is the rank number of the graphs $\overrightarrow{P}_{2^4-1}$, $\overrightarrow{\mathcal{G}}_4$, and $\overleftarrow{\mathcal{G}}_4$. Thus, $\chi_r(\overrightarrow{\mathcal{G}}_4) = \chi_r(\overleftarrow{\mathcal{G}}_4) = \chi_r(\overrightarrow{P}_{2^4-1}) = 4$.