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Rainbow connection number of amalgamation of some graphs

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Abstract

Let G be a nontrivial connected graph. For $k \in \mathbb{N}$, we define a coloring $c : E(G) \to \{1, 2, \ldots, k\}$ of the edges of G such that adjacent edges can be colored the same. A path P in G is a rainbow path if no two edges of P are colored the same. A rainbow path connecting two vertices u and v in G is called rainbow u–v path. A graph G is said rainbow-connected if for every two vertices u and v of G, there exists a rainbow u–v path. In this case, the coloring c is called a rainbow k-coloring of G. The minimum k such that G has a rainbow k-coloring is called the rainbow connection number of G.

For $t \in \mathbb{N}$ and $t \geq 2$, let $\{G_i | i \in \{1, 2, \ldots, t\}\}$ be a finite collection of graphs and each G_i has a fixed vertex v_{oi} called a terminal. The amalgamation $Amal(G_i, v_{oi})$ is a graph formed by taking all the G_i's and identifying their terminals.

We give lower and upper bounds for the rainbow connection number of $Amal(G_i, v_{oi})$ for any connected graph G_i. Additionally, we determine the rainbow connection number of amalgamation of either complete graphs, or wheels, or fans.

Keywords: Amalgamation; Complete graph; Fan; Rainbow connection number; Wheel

1. Introduction

All graphs in this paper are simple, finite, and undirected. The concept of rainbow coloring was introduced by Chartrand et al. [1]. Let G be a nontrivial connected graph. For $k \in \mathbb{N}$, we define a coloring $c : E(G) \to \{1, 2, \ldots, k\}$ of the edges of G such that adjacent edges can be colored the same. A path P in G is a *rainbow path* if no two edges of P are colored the same. A rainbow path connecting two vertices u and v in G is called *rainbow u–v path*. A graph G is said *rainbow-connected* if for every two vertices u and v of G, there exists a rainbow $u - v$ path. In this case, the coloring c is called a *rainbow k-coloring* of G. The minimum k such that G has a rainbow k-coloring is called the *rainbow connection number* $rc(G)$ of G. Clearly $diam(G) \leq rc(G)$ where $diam(G)$ denotes the *diameter* of G.

Chartrand et al. [1] determined the rainbow connection number for some classes of graphs. They showed that $rc(G) = 1$ if and only if G is a complete graph, and $rc(G) = |E(G)|$ if and only if G is a tree. They also determined the rainbow connection number of cycles and wheels. In [2], Syafrizal Sy et al. determined the rainbow connection

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number of fans. The rainbow connection numbers of some other graph classes, namely flower $(C_m; K_n)$ graphs, origami graphs, pizza graphs, n crossed prism graphs, pencil graphs, rocket graphs, and stellar graphs can be seen in [3–7], and [8]. Chakraborty et al. [9] showed that computing the rainbow connection number of a graph is NP-Hard.

There are some results about rainbow connection number of graphs resulted from graph operations. Li and Sun [10] obtained some results on bounds for rainbow connection number of product of graphs, including Cartesian product, composition (lexicographic product), and join of graphs. Basavaraju et al. [11] determined bounds for rainbow connection number of graphs that are obtained by Cartesian product, lexicographic product, strong product, and the operation of taking the power of graph according to the radius of graphs. Gologranc et al. [12] investigated bounds for rainbow connection number on direct, strong, and lexicographic product of graphs. An overview about rainbow connection number can be found in a survey by Li et al. [13] and a book of Li and Sun [14].

2. Main results

The following definition of an amalgamation of graph is taken from [15]. For $t \in \mathbb{N}$ and $i \in \{1, 2, \ldots, t\}$, let G_i be a simple connected graph and $|V(G_i)| = k_i \geq 2$ for some $k_i \in \mathbb{N}$. For $t \geq 2$, let $\{G_1, G_2, \ldots, G_t\}$ be a finite collection of graphs and each $G_i,\ i \in \{1, 2, \ldots, t\}$, has a fixed vertex v_{oi} called a *terminal*. The *amalgamation* $Amal(G_i, v_{oi})$ is a graph formed by taking all the G_i's and identifying their terminals.

Let $G \cong Amal(G_i, v_{oi})$. We denote the identified vertex as v and define $V(G) = \{v\} \cup \{v_{i,j} | 1 \leq i \leq t,\ 1 \leq j \leq k_i - 1\}$.

The following theorem provides a sharp lower and upper bound for the rainbow connection number of amalgamation of arbitrary graphs.

Theorem 2.1. *For $t \in \mathbb{N},\ t \geq 2$, let $\{G_i | i \in \{1, 2, \ldots, t\}\}$ be a finite collection of graphs and each G_i has a fixed vertex v_{oi} called a terminal. If G is the amalgamation of $G_1, G_2, \ldots, G_t$, $Amal(G_i, v_{oi})$, then*

$$diam(G) \leq rc(G) \leq \sum_{i=1}^{t} rc(G_i).$$

Proof. We obtain the lower bound by the fact that $diam(G) \leq rc(G)$. Let c'_i be a rainbow $rc(G_i)$-coloring of G_i. We define a coloring $c : E(G) \to \{1, 2, \ldots, \sum_{i=1}^{t} rc(G_i)\}$ as follows.

$$c(e) = \begin{cases} c'_1(e), & e \in E(G_1); \\ c'_q(e) + \displaystyle\sum_{p=1}^{q-1} rc(G_p), & e \in E(G_q) \text{ for each } q \in \{2, 3, \ldots, t\}. \end{cases}$$

We consider any two vertices $u, w \in V(G)$.

Case 1. $u, w \in V(G_j)$ for $j \in \{1, 2, \ldots, t\}$.

There exists a rainbow u–w path by coloring c corresponding to coloring c'_j.

Case 2. $u \in V(G_j)$ and $w \in V(G_k)$ for some j and k in $\{1, 2, \ldots, t\}$ with $j \neq k$.

There exists a rainbow $u - v$ path and a rainbow $v - w$ path by coloring c corresponding to coloring c'_j and c'_k, respectively, where v is the identified vertex in G corresponding to the terminal v_{oi} in each G_i. We can find a rainbow $u - w$ path by identifying vertex v in a rainbow $u - v$ path and a rainbow $v - w$ path since we use different colors in $V(G_j)$ and $V(G_k)$ by coloring c.

So, c is a rainbow coloring. Thus, $rc(G) \leq \sum_{i=1}^{t} rc(G_i)$. □

In the next two theorems, we prove the existence of an amalgamation graph whose rainbow connection number satisfies either the lower or upper bound in Theorem 2.1.

Theorem 2.2. *Let n and t be two positive integers with $n \geq 3$ and $t \geq 2$. Let $G \cong Amal(G_i, v_{oi})$ where for each $i \in \{1, 2, \ldots, t\}$, G_i is a cycle C_n. For even $n \geq 4$ and $t \geq 2$, or odd $n \geq 3$ and $t = 2$, $rc(G) = diam(G)$.*