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The Fibonacci numbers of certain subgraphs of circulant graphs

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Abstract

The Fibonacci number $\mathscr{F}(G)$ of a graph G with vertex set $V(G)$, is the total number of independent vertex sets $S \subset V(G)$; recall that a set $S \subset V(G)$ is said to be independent whenever for every two different vertices $u, v \in S$ there is no edge between them. In general, the problem to find the Fibonacci number of a graph is NP-complete. Prodinger and Tichy proved that the Fibonacci number of P_n, the path of order n is F_{n+2}, the $n+2$-Fibonacci number; and the Fibonacci number of C_n, the cycle of order n, is L_n, the n-Lucas number.

A circulant graph $C_{n(m_1,m_2,\ldots,m_r)}$ is a graph of order n with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_i v_{i+m_j \pmod n} : i \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, r\}\}$, where $r \in \mathbb{Z}^+$. The values m_j are the jump sizes. In this paper we only deal with the circulant graphs of order n with r consecutive jumps $1, 2, \ldots, r$. $C_{n(1,2,\ldots,r)}$ is denoted by $C_{n_{[r]}}$.

We prove that the total number of independent vertex sets of the family of graphs $C_{n_{[r]}}$ for all $n \geq r + 1$, and for several subgraphs of this family is completely determined by some sequences which are constructed recursively like the Fibonacci and Lucas sequences, even more, these new sequences generalize the Fibonacci and Lucas sequences.

Keywords: Independent set; Generalized Fibonacci numbers; Fibonacci numbers of graphs

1. Introduction

Fibonacci numbers of graphs were introduced by H. Prodinger and R.F. Tichy in 1982 [1]. The total number of independent sets in a graph has been studied by several authors; some results about Fibonacci numbers of trees can be found in [2], and for unicyclic graphs in [3] and [4]. For some results about bounds of independent vertex sets in graphs see [5], and for another general results about Fibonacci numbers of graphs see for example [6] and [7].

In 2000 Kwaśnik and Włoch [8] considered k-independent sets of vertices in graphs. Their principal purpose was to find the total number of k-kernels of P_n and C_n which are specifically k-independent maximal vertex sets [8].

Independently, Morrison-Horton in 2007 [9] found the Fibonacci numbers of the r-powers of paths and cycles by considering the Hopkins and Staton counting method [10].

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For another results about the Fibonacci numbers of graphs that are generalized Fibonacci numbers, see for example [11,12] and [13] and for another application of this generalized Fibonacci sequences see for example [14].

To find the Fibonacci number of a graph G is not a trivial problem, however it is possible to determinate this number if we know the Fibonacci numbers of some particular subgraphs of G.

Let I_G be the set of all independent vertex subsets of G and v a vertex of G, consider the following sets, $I_{G_\in}$ the set of all elements of I_G to which v belongs and let $I_{G_\notin}$ the $I_{G_\in}$ complement set. Since $\left[I_{G_\in}, I_{G_\notin}\right]$ is a partition of I_G, then $|I_G| = \left|I_{G_\in}\right| + \left|I_{G_\notin}\right|$. Observe that there is a bijection between $I_{G_\in}$ and $I_{G-\{v\cup N(v)\}}$, where $G - \{v \cup N(v)\}$ is the graph obtained of G by deleting the vertex v and its neighbors set, the same way, there is a bijection between $I_{G_\notin}$ and I_{G-v}, for this reason, if we know the Fibonacci numbers of the induced subgraphs of G, we can find the Fibonacci number of G by this method, more precisely we will use the following known lemma. Henceforth we will denote by $\mathscr{F}(G)$ the Fibonacci number of G.

Lemma 1. *Let be G a graph and v a vertex of G, then*

$$\mathscr{F}(G) = \mathscr{F}(G - v) + \mathscr{F}(G - [N(v) \cup \{v\}]).$$

$C_{n_{[r]}}$ are graphs with a strong symmetrical structure, so we have some nice properties in this family of graphs; first, observe that for all $r \in \mathbb{Z}^+$, $C_{n_{[r]}}$ is complete whenever $r + 1 \le n \le 2r$; is $2r$-regular for all $n \ge 2r + 1$ and not make sense to talk about circulant graphs $C_{n_{[r]}}$ for $n < r + 1$.

2. The Fibonacci numbers of $C_{n_{[3]}}$ and some of its subgraphs

In order to clarify the general proof, we will work the case $r = 3$. We start by defining some sequences for all $n \in \mathbb{Z}^+$.

Definition 1. The recursive sequence $F_{n_{[3]}}$ is given by

$$F_{n_{[3]}} = \begin{cases} n+1 & \text{if } n \in \{1, \ldots, 4\}. \\ F_{n-1_{[3]}} + F_{n-4_{[3]}} & \text{if } n \ge 5. \end{cases}$$

Definition 2. Let $1 \le k \le 3$, the recursive sequences $F^k_{n_{[3]}}$ are given by:

$$F^k_{n_{[3]}} = \begin{cases} 1 & \text{if } n \in \{1, \ldots, k\}. \\ n+1 & \text{if } n \in \{k+1, \ldots, k+4\}. \\ F_{n-k_{[3]}} + kF_{n-(k+4)_{[3]}} & \text{if } n \ge k+5. \end{cases}$$

Observe the following table:

Table 1
Fibonacci numbers of $C_{n_{[3]}}$ and some of its subgraphs.

n	1	2	3	4	5	6	7	8	9	10	11	12 ...
$F_{n_{[3]}}$	2	3	4	5	7	10	14	19	26	36	50	69 ...
$F^1_{n_{[3]}}$	1	3	4	5	6	9	13	18	24	33	46	64 ...
$F^2_{n_{[3]}}$	1	1	4	5	6	7	11	16	22	29	40	56 ...
$F^3_{n_{[3]}}$	1	1	1	5	6	7	8	13	19	26	34	47 ...

The sequences that we just described have a strong intrinsic relationship between one another under some properties; consider the nth and $(n + 4)$th terms of the first sequence, note that their sum is just the $(n + 5)$th term of the second sequence, there is $F_{n_{[3]}} + F_{n+4_{[3]}} = F^1_{n+5_{[3]}}$, this for all $n \in \mathbb{Z}^+$, in the same way $F_{n_{[3]}} + F^1_{n+5_{[3]}} = F^2_{n+6_{[3]}}$ and $F_{n_{[3]}} + F^2_{n+6_{[3]}} = F^3_{n+7_{[3]}}$. Now we are going to prove that this property is true.