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Independent and monochromatic absorbent sets in infinite digraphs☆

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Abstract

Let D be a digraph, we say that it is an m-coloured digraph if the arcs of D are coloured with at most m-colours. An (u, v) arc is symmetrical if (v, u) is also an arc of D. A directed path (resp. directed cycle) is monochromatic if all of its arcs are coloured with the same colour, and it is quasi-monochromatic if at most one of its arcs is coloured with different colour.

A set $AN \subset V(D)$ is an $\mathcal{A}$-kernel if it satisfies the following conditions:

(1) For every pair of vertices $x, y \in AN$, there is no arc between them, we say that AN is independent.
(2) For every $x \notin AN$, there exists an xy-monochromatic path for some $y \in AN$, it means AN is absorbent by monochromatic paths.

An infinite sequence of different vertices $(x_1, x_2, x_3, \dots)$ such that $(x_i, x_{i+1}) \in A(D)$ for every $i \in \mathbb{N}$ will be called an infinite outward path.

In this paper we introduce the definitions of $\mathcal{A}$-kernel and $\mathcal{A}$-semikernel, but also we prove the following theorem: Let D be a possibly infinite digraph, if every directed cycle and every infinite outward path has two consecutive vertices, say x_i and x_{i+1}, such that there exists an $x_{i+1}x_i$-monochromatic path, then D has an $\mathcal{A}$-kernel.

Keywords: Independence; Monochromatic absorbance; Infinite digraph; $\mathcal{A}$-kernel

1. Introduction

Let D be a digraph, denote the vertices of D by $V(D)$ and the arcs of D by $A(D)$. An arc (u, v) in D is called **symmetrical** if both ordered pairs (u, v) and (v, u) are arcs of D. A set $I \subset V(D)$ is **independent** if no two vertices in I are adjacent.

A digraph D is m-**coloured** if any arc of D have assigned just one colour from m different colours. A **monochromatic path** (resp. **monochromatic cycle**) is a directed path (directed cycle) in an m-coloured digraph such that all of

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its arcs are coloured alike. Also a **quasi-monochromatic path** (resp. **quasi-monochromatic cycle**) is a directed path (directed cycle) in which all of its arcs are coloured alike except for at most one arc.

Let D be an m-coloured digraph, $A \subset V(D)$ is **absorbent by monochromatic paths** if for every $x \in V(D) - A$ there exists an xy-monochromatic path with $y \in A$. And $S \subset V(D)$ is **semi-absorbent by monochromatic paths** if it satisfies; if there exists an arc $(y, x) y \in S$ and $x \notin S$, then there exists an xw-monochromatic path for some $w \in S$.

Definition 1. A set $AN \subset V(D)$ is $\mathcal{A}$-**kernel** if it is absorbent by monochromatic paths and independent. In the same way a set $M \subset V(D)$ is $\mathcal{A}$-**semikernel** if it is semiabsorbent by monochromatic paths and independent.

This definition is obtained by the search of a kernel generalization, in which are considered different conditions of kernels and kernels by monochromatic paths. The idea of mixing different conditions was also analysed in [1], where they considered the vertices of a digraph D to colour the arcs of a digraph G, and use the arcs of D to determine when a sequence of colours is valid (not only monochromatic).

The kernels and the kernels by monochromatic paths have been studied over the years looking for conditions in the digraph that makes certain the existence of both objects, these conditions are very diverse, like the number of colours used for colouring the digraph [2], the number or the sequence of colours in certain cycles or similar structures [3–5], and conditions over the structure of the digraph, for example: transitive digraphs, tournaments, etc. [6–8].

Now, let us remark some relations between kernels, $\mathcal{A}$-kernels and kernels by monochromatic paths:

Remark 2. (1) Every digraph with kernel satisfies that any colouration of D has an $\mathcal{A}$-kernel. But not every digraph with $\mathcal{A}$-kernel has a kernel.

(2) Every m-coloured digraph D that has a kernel by monochromatic paths also has an $\mathcal{A}$-kernel. But not every m-coloured digraph with $\mathcal{A}$-kernel has a kernel by monochromatic paths, see Fig. 2.

Almost all of the historical conditions for kernels and kernels by monochromatic paths are for finite digraphs, in the case of infinite digraphs the main concern is the possible existence of certain structures like the infinite outward paths, that is, an infinite sequence of different vertices $(x_1, x_2, x_3, \ldots)$ such that $(x_i, x_{i+1}) \in A(D)$ for every $i \in \mathbb{N}$.

In this paper we consider digraphs without arcs of the form (x, x), actually these do not generate any problem and the results still work for digraphs with these kind of arcs. We just avoid them for the concept of independence used for different authors.

We prove that if D is an m-coloured digraph, such that every cycle in D and every infinite outward path have two consecutive vertices x_i, x_{i+1} such that there exists an $x_{i+1}x_i$-monochromatic path, then D has an $\mathcal{A}$-kernel.

We will need the following lemma, which is a classic result in the Digraph Theory.

Lemma 3. *Every closed directed walk contains a directed cycle.*

2. The main result

With these remarks now we can start with the main theorem.

Theorem 4. *Let D be an m-coloured digraph, such that every cycle in D and every infinite outward path has two consecutive vertices x_i, x_{i+1} such that there exists an $x_{i+1}x_i$-monochromatic path, then D has an $\mathcal{A}$-kernel.*

Proof. First we will see that D always has an $\mathcal{A}$-semikernel, even more, it has an $\mathcal{A}$-semikernel consisting of just one vertex.

Suppose this is not true, it means, for any $x \in V(D)$, $\{x\}$ is not an $\mathcal{A}$-semikernel. Since any set with just one vertex is always independent, then it follows that for any $x \in V(D)$ there exists $y \in V(D)$ such that $(x, y) \in A(D)$ and there is no yx-monochromatic path.

Take $x_1 \in V(D)$, then there exists $x_2 \in V(D)$ such that $(x_1, x_2) \in A(D)$ and there is no x_2x_1-monochromatic path, also there exists $x_3 \in V(D)$ such that $(x_2, x_3) \in A(D)$ and there is no x_3x_2-monochromatic path, and subsequently in this way there exists a sequence of vertices $x_1, x_2, x_3, x_4, \ldots$ which satisfies that $(x_i, x_{i+1}) \in A(D)$ and there is no x_ix_{i-1}-monochromatic path, even more this sequence generate an infinite directed walk, $(x_1, x_2, x_3, x_4, \ldots)$ in D.

Case (I) If in this infinite directed walk all of the vertices are different, then this will be an infinite outward path, which by hypothesis has two consecutive vertices x_i, x_{i+1} such that there exists an $x_{i+1}x_i$-monochromatic path, but this is a contradiction with the election of the vertices of the sequence.