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Even harmonious labelings of disjoint graphs with a small component

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Abstract

A graph G with q edges is said to be harmonious if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y) \pmod q$, the resulting edge labels are distinct. If G is a tree, exactly one label may be used on two vertices. Over the years, many variations of harmonious labelings have been introduced.

We study a variant of harmonious labeling. A function f is said to be a properly even harmonious labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to $2(q-1)$ and the induced function f^* from the edges of G to $0, 2, \ldots, 2(q-1)$ defined by $f^*(xy) = f(x) + f(y) \pmod{2q}$ is bijective. We investigate the existence of properly even harmonious labelings of families of disconnected graphs with one of C_3, C_4, K_4 or W_4 as a component.

Keywords: Properly even harmonious labelings; Even harmonious labelings; Harmonious labelings; Graph labelings

1. Introduction

A vertex *labeling* of a graph G is a mapping f from the vertices of G to a set of elements, often integers. Each edge xy has a label that depends on the vertices x and y and their labels $f(x)$ and $f(y)$. Graph labeling methods began with Rosa [1] in 1967. In 1980, Graham and Sloane [2] introduced harmonious labelings in connection with error-correcting codes and channel assignment problems. There have been three published papers on even harmonious graph labelings by Sarasija and Binthiya [3,4] and Gallian and Schoenhard [5]. In [6] we focus on the existence of properly even harmonious labelings for the disjoint union of cycles and stars, unions of cycles with paths, unions of squares of paths, and unions of paths. In this paper we investigate the existence of properly even harmonious labelings of families of disconnected graphs with one of C_3, C_4, K_4 or W_4 as a component.

An extensive survey of graph labeling methods is available online [7]. We follow the notation in [7].

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2. Preliminaries

Definition 2.1. A graph G with q edges is said to be *harmonious* if there exists an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y) \pmod q$, the resulting edge labels are distinct. When G is a tree, exactly one edge label may be used on two vertices.

Definition 2.2. A function f is said to be an *even harmonious* labeling of a graph G with q edges if f is an injection from the vertices of G to the integers from 0 to $2q$ and the induced function f^* from the edges of G to $0, 2, \ldots, 2(q-1)$ defined by $f^*(xy) = f(x) + f(y) \pmod{2q}$ is bijective.

Because 0 and $2q$ are equal modulo $2q$, Gallian and Schoenhard [5] introduced the following more desirable form of even harmonious labelings.

Definition 2.3. An even harmonious labeling of a graph G with q edges is said to be a *properly even harmonious labeling* if the vertex labels belong to $\{0, 2, \ldots, 2q-2\}$.

Definition 2.4. A graph that has a (properly) even harmonious labeling is called *(properly) even harmonious* graph.

Bass [8] has observed that for connected graphs, a harmonious labeling of a graph with q edges yields an even harmonious labeling by multiplying each vertex label by 2 and adding the vertex labels modulo $2q$. Gallian and Schoenhard [5] showed that for any connected even harmonious labeling, we may assume the vertex labels are even. Therefore, for a connected graph we can obtain a harmonious labeling from a properly even harmonious labeling by dividing each vertex label by 2 and adding the vertex labels modulo q. Consequently, we focus our attention on disconnected graphs.

3. Disconnected graphs

Definition 3.1. We define an *odd hairy cycle* as an odd cycle with one or more pendant edges attached.

Definition 3.2. We call a graph G *pseudo-bipartite* if G is not bipartite but the removal of one edge results in a bipartite graph.

We will use ${C_m}^{+n}$ to denote an m-cycle with n pendant edges attached.

To describe our labeling of ${C_m}^{+n}$ for m odd, we draw the m-cycle in a zigzag fashion as shown in Fig. 1. The pendant edges incident to the cycle vertices are drawn so that the endpoints are on the side opposite the cycle vertices. Ignoring the edge that joins the first and last vertices of the odd cycle, we have a bipartite graph with one partite set on the left (L) and the other on the right (R). We call this a *pseudo-bipartite* graph (Definition 3.2). Denote the number of edges in L as l and the number of edges in R as r.

It is convenient to denote an odd hairy cycle by specifying the sizes l and r of pseudo-bipartite sets L and R as ${C_m}^{+n}(l, r)$.

Theorem 3.1. *$C_4 \cup {C_m}^{+n}(l, r)$ is properly even harmonious.*

Proof. The modulus is $2m + 2n + 8$.

Arrange the pseudo-bipartite sets as described above and shown in Fig. 1. Label L of ${C_m}^{+n}(l, r)$ as $0, 2, \ldots, 2l-2$. Label R continuing with $2l, 2l+2, \ldots, 2l+2r-2$. The corresponding edge labels are $2l-2, 2l, \ldots, 4l+2r-4$.

Label the vertices of C_4 consecutively as $2m+2n+7, 4l+2r-1, 3, 4l+2r+1$. The corresponding edge labels are $4l+2r-2, 4l+2r+2, 4l+2r+4, 4l+2r$. See Fig. 1.

To verify there are no duplicate vertex labels in the C_4 component, notice that $2m + 2n + 7 < 2m + 2n + 11$ and $4l + 2r - 1 < 4l + 2r + 1$. Since $4l + 2r = 4$ implies that $l = 1$ and $r = 0$, and likewise $4l + 2r = 2$ implies that $l = 0$ and $r = 1$, we know there is no duplication of labels on the C_4 component. For the ${C_m}^{+n}(l, r)$ component, notice that the vertex labels are all increasing with common difference of 2. The largest gap between vertex labels is less than the modulus so there is no wrap around. □