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Acyclicity in edge-colored graphs

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ABSTRACT

A walk W in edge-colored graphs is called properly colored (PC) if every pair of consecutive edges in W is of different color. We introduce and study five types of PC acyclicity in edge-colored graphs such that graphs of PC acyclicity of type i is a proper superset of graphs of acyclicity of type $i+1$, $i = 1, 2, 3, 4$. The first three types are equivalent to the absence of PC cycles, PC closed trails, and PC closed walks, respectively. While graphs of types 1, 2 and 3 can be recognized in polynomial time, the problem of recognizing graphs of type 4 is, somewhat surprisingly, NP-hard even for 2-edge-colored graphs (i.e., when only two colors are used). The same problem with respect to type 5 is polynomial-time solvable for all edge-colored graphs. Using the five types, we investigate the border between intractability and tractability for the problems of finding the maximum number of internally vertex-disjoint PC paths between two vertices and the minimum number of vertices to meet all PC paths between two vertices.

1. Introduction

A *walk* in a multigraph is a sequence $W = v_1e_1v_2 \ldots v_{p-1}e_{p-1}v_p$ of alternating vertices and edges such that vertices v_i and v_{i+1} are end-vertices of edge e_i for every $i \in [p-1]$. A walk W is *closed* (*open*, respectively) if $v_1 = v_p$ ($v_1 \neq v_p$, respectively). A *trail* is a walk in which all edges are distinct, a *path* is a non-closed walk in which all vertices are distinct, and a *cycle* is a closed walk where all vertices apart from the first and last ones are distinct.

In this paper, we study properly colored walks in graphs with colored edges, which are called *edge-colored graphs* or *c-edge-colored graphs* when colors are taken from the set $[c] = \{1, 2, \ldots, c\}$. For 2-edge-colored graphs we use colors blue and red instead of 1 and 2. A walk $W = v_1e_1v_2 \ldots v_{p-1}e_{p-1}v_p$ is *properly colored (PC)* if edges e_i and e_{i+1} are of different colors for every $i \in \{1, 2, \ldots, p-2\}$ and, in addition, if W is closed then edges e_{p-1} and e_1 are of different colors. PC walks are of interest in graph theory applications, e.g., in molecular biology [5–7,17] and in VLSI for compacting programmable logical arrays [12]. They are also of interest in graph theory itself as generalizations of walks in undirected and directed graphs. Indeed, consider the standard transformation from a directed graph D into a 2-edge-colored graph G by replacing every arc uv of D by a path with blue edge uw_{uv} and red edge $w_{uv}v$, where w_{uv} is a new vertex [3]. Clearly, every directed walk in D corresponds to a PC walk in G and vice versa. On the other hand, if every edge has a distinct color (or more generally, if the coloring is proper), then clearly all trails of the underlying undirected graph G are PC trails. There is an extensive

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literature on PC walks: for a detailed survey of pre-2009 publications, see Chapter 16 of [3], and more recent papers include [2,9,11,13–15].

The following notion of a monochromatic vertex will often be used in this paper. A vertex v in an edge-colored graph G is called *G-monochromatic* if all edges incident to v in G are of the same color. Clearly, a PC closed walk has no G-monochromatic vertex.

It is well-known and trivial to prove that every undirected and directed graph with no cycles, has no closed walks either. Surprisingly, this is not the case for PC cycles and PC walks. In fact, the properties of having no PC cycles, having no PC closed trails, and having no PC closed walks, are all distinct. In this paper, in order to better understand the structure of acyclic edge-colored graphs, we introduce five types of PC acyclicity as follows.

Definition 1. Let G be a c-edge-colored undirected graph, $c \geq 2$. An ordering $v_1, v_2, \ldots, v_n$ of vertices of G is **of type**

1. if for every $i \in [n]$, all edges from v_i to each connected component of $G[\{v_{i+1}, v_{i+2}, \ldots, v_n\}]$ have the same color;
2. if for every $i \in [n]$, all edges from v_i to $\{v_{i+1}, v_{i+2}, \ldots, v_n\}$ which are not bridges in $G[\{v_i, v_{i+1}, \ldots, v_n\}]$ have the same color.
3. if for every $i \in [n]$, all edges from v_i to $\{v_{i+1}, v_{i+2}, \ldots, v_n\}$ have the same color;
4. if for every $i \in [n]$, all edges from v_i to $\{v_{i+1}, v_{i+2}, \ldots, v_n\}$ have the same color and all edges from v_i to $\{v_1, v_2, \ldots, v_{i-1}\}$ have the same color;
5. if for every $i \in [n]$, all edges from v_i to $\{v_{i+1}, v_{i+2}, \ldots, v_n\}$ have the same color and all edges from v_i to $\{v_1, v_2, \ldots, v_{i-1}\}$ have the same color but different from the color of edges from v_i to $\{v_{i+1}, v_{i+2}, \ldots, v_n\}$.

Definition 2. Let $i \in [5]$. G is **PC acyclic of type** i if it has an ordering $v_1, v_2, \ldots, v_n$ of vertices of type i.

Clearly, the class of c-edge-colored acyclic graphs of type i contains the class of c-edge-colored acyclic graphs of type $i+1$, $i \in [4]$. We will see later in the paper that the containments are proper. We will see that graphs of the first two types coincide with edge-colored graphs without PC cycles and without PC closed trails, respectively. These two classes of edge-colored graphs were characterized by Yeo [19] and Abouelaoualim et al. [1], respectively. We will prove that graphs of PC acyclicity of type 3 are edge-colored graphs without PC walks. We are unaware of a "nice" characterization of edge-colored graphs of type 4. In fact, we show that it is NP-hard to recognize graphs of this type, which is somewhat surprising as we prove that recognition of all other types is polynomial-time solvable. We will prove that an edge-colored graph is acyclic of type 5 if and only if every vertex is incident to edges of at most two colors and every cycle C has a positive even number of vertices incident, in C, to edges of the same color. For 2-edge-colored graphs this is equivalent to being bipartite with no PC cycle. Therefore for 2-edge-colored graphs, being PC acyclic of type 5 is the same as being bipartite and PC acyclic of type 1.

Using the five types, we will investigate the border between intractability and tractability for the problems of finding the maximum number of internally vertex-disjoint PC paths between two vertices and of finding the minimum number of vertices to eliminate all PC paths between two vertices. We will prove that both problems are NP-hard for 2-edge-colored graphs of PC acyclicity of type 3 (and thus of types 1 and 2), but polynomial time solvable for edge-colored graphs of PC acyclicity of type 4 (and 5). We will also show that while Menger's theorem does not hold in general, even on 2-edge-colored graphs of PC acyclicity of type 3 (or 1 or 2), it holds on edge-colored graphs of PC acyclicity of type 4 (and 5).

The rest of the paper is organized as follows. In Section 2, we study the five types of acyclicity of edge-colored graphs. Section 3 is devoted to PC paths and separators in edge-colored graphs. Finally, in Section 4 we discuss an open problem.

2. Types of PC acyclic edge-colored graphs

In this section, we study the five types of PC acyclicity introduced in the previous section.

The fact that a c-edge-colored graph G is PC acyclic of type 1 if and only if G has no PC cycle follows immediately from a theorem by Yeo [19] (a special case of Yeo's theorem for $c = 2$ was obtained by Grossman and Häggkvist [10]).

Theorem 1. *If a c-edge-colored graph G has no PC cycle then G has a vertex z such that every connected component of $G - z$ is joined to z by edges of the same color.*

We will prove the following easy consequence of Theorem 1.

Corollary 1. *A c-edge-colored graph G is PC acyclic of type 1 if and only if G has no PC cycle.*

Proof. Suppose that G is PC acyclic of type 1 and has a PC cycle C. Consider an acyclic ordering of $V(G)$ of type 1. Let x be the vertex on C with lowest subscript in the acyclic ordering. Observe that $C - x$ must belong to the same component in $G - x$ and all vertices of $C - x$ come after x in the acyclic ordering. Thus, x must have both incident edges in the cycle of the same color, a contradiction.

Now let G have no PC cycle. By Theorem 1, G has a vertex z such that every connected component of $G - z$ is joined to z by edges of the same color. Set $v_1 = z$ and consider $G - z$ to obtain $v_2, \ldots, v_n$. Clearly, the resulting ordering is PC acyclic of type 1. $\square$