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Nonexistence of embeddings with uniformly bounded distortions of Laakso graphs into diamond graphs

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ABSTRACT

Diamond graphs and Laakso graphs are important examples in the theory of metric embeddings. Many results for these families of graphs are similar to each other. In this connection, it is natural to ask whether one of these families admits uniformly bilipschitz embeddings into the other. The well-known fact that Laakso graphs are uniformly doubling but diamond graphs are not, immediately implies that diamond graphs do not admit uniformly bilipschitz embeddings into Laakso graphs. The main goal of this paper is to prove that Laakso graphs do not admit uniformly bilipschitz embeddings into diamond graphs.

1. Introduction

Diamond graphs and Laakso graphs are important examples in the theory of metric embeddings, see [2,7–14] . Many results for these families of graphs are similar to each other, see the example after Definitions 1.1 and 1.2 and in Section 3. In this connection, the question emerges: does one of these families admit uniformly bilipschitz embeddings into the other? The well-known fact that Laakso graphs are uniformly doubling but diamond graphs are not uniformly doubling – see Definition 1.3 and the subsequent discussion – immediately implies that diamond graphs do not admit uniformly bilipschitz embeddings into Laakso graphs. The main goal of this paper is to prove that Laakso graphs do not admit uniformly bilipschitz embeddings into diamond graphs.

To the best of our knowledge, the first paper in which diamond graphs $\{D_n\}_{n=0}^\infty$ were used in Metric Geometry is the conference version of [4], which was published in 1999.

Definition 1.1. Diamond graphs $\{D_n\}_{n=0}^\infty$ are defined recursively: The *diamond graph* of level 0 has two vertices joined by an edge of length 1 and is denoted by D_0. The *diamond graph* D_n is obtained from D_{n-1} in the following way. Given an edge $uv \in E(D_{n-1})$, it is replaced by a quadrilateral u, a, v, b, with edges ua, av, vb, bu. (See Fig. 1.)

Two different normalizations of the graphs $\{D_n\}_{n=1}^\infty$ can be found in the literature:

- *Unweighted diamonds:* Each edge has length 1.
- *Weighted diamonds:* Each edge of D_n has length 2^{-n}.

In both cases, we endow the vertex sets of $\{D_n\}_{n=0}^\infty$ with their shortest path metrics.

For weighted diamonds, the identity map $D_{n-1} \mapsto D_n$ is an isometry and, in this case, the union of D_n endowed with the metric induced from $\{D_n\}_{n=0}^\infty$ is called the *infinite diamond* and denoted by D_ω.

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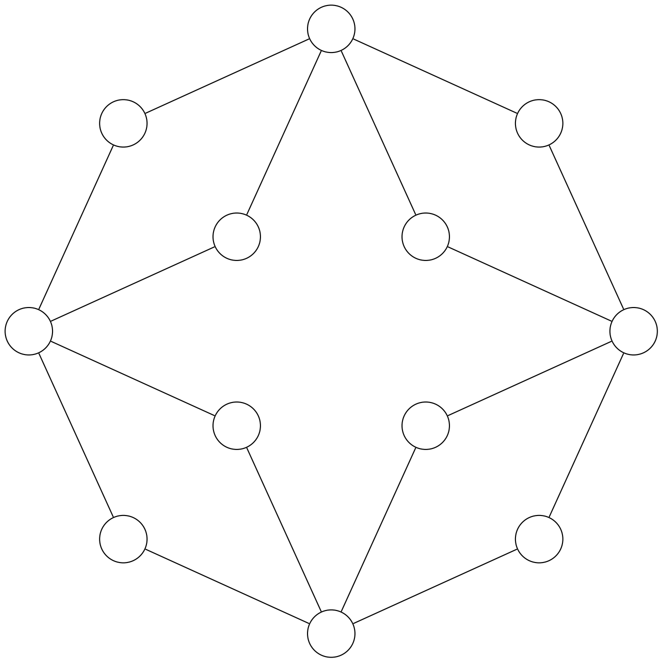

Fig. 1. Diamond D_2.

Another family of graphs considered in the present article is that of Laakso graphs. The Laakso graphs were introduced in [8], but they were inspired by the construction of Laakso [7].

Definition 1.2. Laakso graphs $\{L_n\}_{n=0}^{\infty}$ are defined recursively: The *Laakso graph* of level 0 has two vertices joined by an edge of length 1 and is denoted L_0. The *Laakso graph* L_n is obtained from L_{n-1} according to the following procedure. Each edge $uv \in E(L_{n-1})$ is replaced by the graph L_1 exhibited in Fig. 2, the vertices u and v are identified with the vertices of degree 1 of L_1.

Similarly to the case of diamond graphs, the two different normalizations of the graphs $\{L_n\}_{n=1}^{\infty}$ are used:

- *Unweighted Laakso graphs:* Each edge has length 1.
- *Weighted Laakso graphs:* Each edge of L_n has length 4^{-n}.

In both situations, we endow vertex sets of $\{L_n\}_{n=0}^{\infty}$ with their shortest path metrics. In the case of weighted Laakso graphs, the identity map $L_{n-1} \mapsto L_n$ is an isometry and the union of L_n, endowed with the metric induced from $\{L_n\}_{n=0}^{\infty}$, is called the *Laakso space* and denoted by L_ω.

Many known results for one of the aforementioned families admit analogues for the other. For example, it is known [6] that both of the families can be used to characterize superreflexivity. Further, the two families consist of planar graphs with poor embeddability into Hilbert space, see [7,8,11]. Moreover, in many situations, the proofs used for one of the families can be easily adjusted to work for the other family. For instance, this is the case for the Markov convexity (see [10, Section 3] and Section 3 of this paper). There are similarities between results for the infinite diamond and the Laakso space, too. For example, neither of spaces D_ω and L_ω admits bilipschitz embeddings into any Banach space with the Radon–Nikodým property [3,12].

On the other hand, the families $\{D_n\}$ and $\{L_n\}$ are not alike in some important metrical respects, and the corresponding properties of the Laakso graphs were among the reasons for their introduction. To exemplify the differences, it can be mentioned that the Laakso graphs are uniformly doubling (see [8, Theorem 2.3]) and unweighted Laakso graphs have uniformly bounded geometry, whereas diamond graphs do not possess any of these properties. These facts are well known. Nevertheless, for the convenience of the readers, they will be proved after recalling the necessary definitions as a suitable reference is not available.

Definition 1.3. (i) A metric space X is called *doubling* if there is a constant $D \geq 1$ such that every bounded set B in X can be covered by at most D sets of diameter $\mathrm{diam}(B)/2$. A sequence of metric spaces is called *uniformly doubling* if all of them are doubling and the constant D can be chosen to be the same for all of them.

(ii) A metric space X is said to have *bounded geometry* if there is a function $M : (0, \infty) \to (0, \infty)$ such that each ball of radius r in X has at most $M(r)$ elements. A sequence of metric spaces is said to have *uniformly bounded geometry* if the function $M(r)$ can be chosen to be the same for all of them.