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Low 5-stars in normal plane maps with minimum degree 5

O.V. Borodin [a], A.O. Ivanova [b,*]

[a] Sobolev Institute of Mathematics, Novosibirsk 630090, Russia

[b] Ammosov North-Eastern Federal University, Yakutsk, 677000, Russia

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ABSTRACT

In 1996, Jendrol' and Madaras constructed a plane triangulation with minimum degree 5 in which the minimum vertex degree $h(S_5)$ of 5-stars is arbitrarily large. This construction has minor $(5, 5, 5, 5)$-stars, that is 5-vertices with four 5-neighbors. It has been open if forbidding minor $(5, 5, 5, 5)$-stars makes $h(S_5)$ finite.

We prove that every normal plane map with minimum degree 5 and no minor $(5, 5, 5, 5)$-stars satisfies $h(S_5) \le 13$ and construct such a map with $h(S_5) = 12$.

1. Introduction

A normal plane map (NPM for short) is a connected plane pseudograph in which loops and multiple edges are allowed, but the degree of each vertex and face is at least three. The degree of a vertex v is denoted by $d(v)$. A *k-vertex* is a vertex v with $d(v) = k$. A k^+*-vertex* (k^-*-vertex*) has degree at least k (at most k). Similar notation is used for faces.

A k-star $S_k(v)$ with k rays is *minor* if its center v has degree (in the NPM) at most 5. We consider both minor and *free* stars, where the latter have no restrictions on the degree of their central vertex.

An $S_k(v)$ is a $(d_1, \dots, d_k)$*-star* if $d(v_i) \le d_i$, $1 \le i \le k$, where v_i's are neighbors of v in any order.

The class of NPMs with minimum degree at least 5 is denoted by $\mathbf{M}_5$. An arbitrary NPM in $\mathbf{M}_5$ is denoted by M_5.

The *weight* of a subgraph of an NPM is the degree-sum of its vertices. By $w(S_k^{(m)})$ and $w(S_k)$ denote the minimum weight of minor and free k-stars, respectively, in a given NPM. By definition, $w(S_k) \le w(S_k^{(m)})$. For the minimum weight of edges, which are 1-stars, and that of 3-paths, which are 2-stars, we also use the usual notation w_2 and w_3, respectively.

The *height* of a subgraph of an NPM is the maximum degree of its vertices. By $h(S_k^{(m)})$ and $h(S_k)$ we denote the minimum height of minor and free k-stars, respectively, in a given NPM. Clearly, $h(S_k) \le h(S_k^{(m)})$.

In 1904, Wernicke [29] proved that every M_5 has a 5-vertex adjacent to a 6^--vertex, that is a minor (6)-star. This was strengthened by Franklin [17] in 1922 to the existence of a minor (6, 6)-star, where all bounds are sharp.

In 1940, Lebesgue [27, p. 36] gave an approximate description of the neighborhoods of 5-vertices in $\mathbf{M}_5$. In particular, this description implies the above mentioned results by Wernicke [29] and Franklin [17] and shows that there is a 5-vertex with three 8^--neighbors. Another corollary of Lebesgue's description [27] is that $w(S_3^m) \le 24$, which was improved in 1996 by Jendrol' and Madaras [19] to the sharp bound $w(S_3^m) \le 23$. Furthermore, Jendrol' and Madaras [19] gave a precise description of minor 3-stars in $\mathbf{M}_5$: there is a (6, 6, 6)- or (5, 6, 7)-star.

* Corresponding author.
E-mail addresses: brdnoleg@math.nsc.ru (O.V. Borodin), shmgnanna@mail.ru (A.O. Ivanova).

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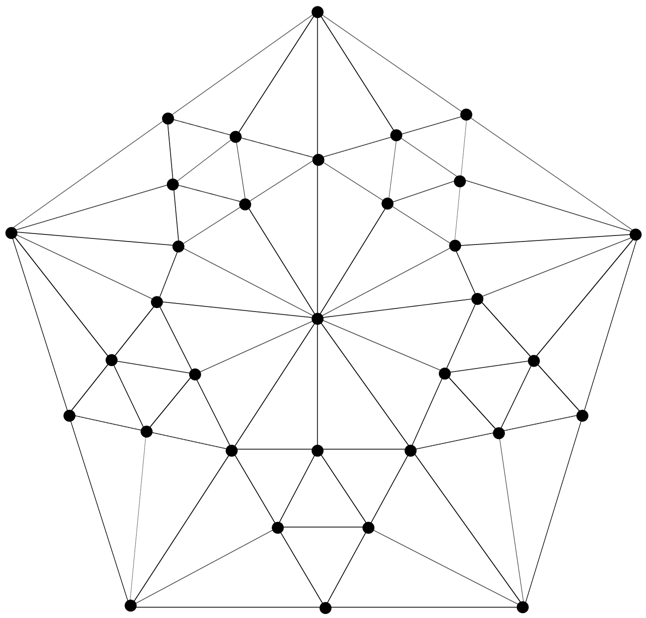

Fig. 1. A construction with $h(S_5) = 12$ and no minor $(5, 5, 5, 5)$-stars.

Also, Lebesgue [27] proved $w(S_4^{(m)}) \leq 31$, which was strengthened by Borodin and Woodall [8] to the sharp bound $w(S_4^{(m)}) \leq 30$. Note that $w(S_3^{(m)}) \leq 23$ easily implies $w(S_2^{(m)}) \leq 17$ and immediately follows from $w(S_4^{(m)}) \leq 30$ (in both cases, it suffices to delete a vertex of maximum degree from a minor star of minimum weight). In [12], we obtained a tight description of minor 4-stars in $\mathbf{M}_5$.

As for minor 5-stars in $\mathbf{M}_5$, it follows from Lebesgue [27, p. 36] that if there are no minor $(5, 5, 6, 6)$-stars, then $w(S_5^{(m)}) \leq 68$ and $h(S_5^{(m)}) \leq 41$. Borodin, Ivanova, and Jensen [13] showed that the presence of minor $(5, 5, 6, 6)$-stars can make $w(S_5^{(m)})$ arbitrarily large and otherwise lowered Lebesgue's bounds to $w(S_5^{(m)}) \leq 55$ and $h(S_5^{(m)}) \leq 28$. On the other hand, a construction in [13] shows that $w(S_5^{(m)}) \geq 48$ and $h(S_5^{(m)}) \geq 20$. Recently, we have proved [15] that $w(S_5^{(m)}) \leq 51$ and $h(S_5^{(m)}) \leq 23$.

For the height of free 5-stars, the presence of minor $(5, 5, 5, 5)$-stars turns out to be crucial. Back in 1996, Jendrol' and Madaras [19] constructed an M_5 with $h(S_5) = \infty$, and their construction has minor $(5, 5, 5, 5)$-stars.

It has been open whether forbidding minor $(5, 5, 5, 5)$-stars makes $h(S_5)$ finite. The purpose of our paper is to prove that $h(S_5) \leq 13$ and construct such a map with $h(S_5) = 12$.

Theorem 1. *Every normal plane map with minimum degree* 5 *and no minor* $(5, 5, 5, 5)$*-stars has a* 5*-star of height at most* 13 *and there is such a map with height* 12.

Problem 2. *Is it true that every normal plane map with minimum degree* 5 *and no minor* $(5, 5, 5, 5)$*-stars has a* 5*-star of height at most* 12?

More results on the structure of edges and higher stars in various classes of NPMs can be found in [1–7,9–11,16,18,20–26,28], with a detailed summary in [14].

2. Proof of Theorem 1

In Fig. 1, we see an insertion to every face of the dodecahedron that results in a plane triangulation with $\delta = 5, h(S_5) = 12$, and no minor $(5, 5, 5, 5)$-stars.

Now let M' be a counterexample to the upper bound in Theorem 1. In particular, every 13^--vertex v in M' has at least $d(v) - 4$ neighbors of degree at least 14, and every 5-vertex has at most three 5-neighbors.

Let M be a counterexample with the most edges on $V(M')$.

Remark 3. No 14^+-vertex v in M is incident with a 4^+-face $f = \cdots xwv$. Indeed, adding the diagonal xv to f cannot create a minor $(5, 5, 5, 5)$-star or a 5-star consisting of 13^--vertices.