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Factorizations of complete multipartite hypergraphs

M.A. Bahmanian

Department of Mathematics, Illinois State University, Normal, IL 61790-4520, USA

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ABSTRACT

In a mathematics workshop with mn mathematicians from n different areas, each area consisting of m mathematicians, we want to create a collaboration network. For this purpose, we would like to schedule daily meetings between groups of size three, so that (i) two people of the same area meet one person of another area, (ii) each person has exactly r meeting(s) each day, and (iii) each pair of people of the same area have exactly λ meeting(s) with each person of another area by the end of the workshop. Using hypergraph amalgamation–detachment, we prove a more general theorem. In particular we show that above meetings can be scheduled if: $3 \mid rm$, $2 \mid rnm$ and $r \mid 3\lambda(n-1)\binom{m}{2}$. This result can be viewed as an analogue of Baranyai's theorem on factorizations of complete multipartite hypergraphs.

1. Introduction

Throughout this paper, $\mathbb{N}$ is the set of positive integers, $m, n, r, \lambda \in \mathbb{N}$, and $[n] := \{1, \dots, n\}$. In a mathematics workshop with mn mathematicians from n different areas, each area consisting of m mathematicians, we want to create a collaboration network. For this purpose, we would like to schedule daily meetings between groups of size three, so that (i) two people of the same area meet one person of another area, (ii) each person has exactly r meeting(s) each day, and (iii) each pair of people of the same area have exactly λ meeting(s) with each person of another area by the end of the workshop. Using hypergraph amalgamation–detachment, we prove a more general theorem. In particular we show that above meetings can be scheduled if: $3 \mid rm$, $2 \mid rnm$ and $r \mid 3\lambda(n-1)\binom{m}{2}$.

A *hypergraph* $\mathcal{G}$ is a pair (V, E) where V is a finite set called the vertex set, E is the edge multiset, where every edge is itself a multi-subset of V. This means that not only can an edge occur multiple times in E, but also each vertex can have multiple occurrences within an edge. The total number of occurrences of a vertex v among all edges of E is called the *degree*, $d_{\mathcal{G}}(v)$ of v in $\mathcal{G}$. For $h \in \mathbb{N}$, $\mathcal{G}$ is said to be *h-uniform* if $|e| = h$ for each $e \in E$. For $r, r_1, \dots, r_k \in \mathbb{N}$, an r-factor in a hypergraph $\mathcal{G}$ is a spanning r-regular sub-hypergraph, and an $(r_1, \dots, r_k)$*-factorization* is a partition of the edge set of $\mathcal{G}$ into $F_1, \dots, F_k$ where F_i is an r_i-factor for $i \in [k]$. We abbreviate $(r, \dots, r)$-factorization to r-factorization.

The hypergraph $K_n^h := (V, \binom{V}{h})$ with $|V| = n$ (by $\binom{V}{h}$ we mean the collection of all h-subsets of V) is called a *complete h*-uniform hypergraph. In connection with Kirkman's schoolgirl problem [14], Sylvester conjectured that K_n^h is 1-factorable if and only if $h \mid n$. This conjecture was finally settled by Baranyai [8]. Let $\mathscr{K}_{n\times m}^3$ denote the 3-uniform hypergraph with vertex partition $\{V_i : i \in [n]\}$, so that $V_i = \{x_{ij} : j \in [m]\}$ for $i \in [n]$, and with edge set $E = \{\{x_{ij}, x_{ij'}, x_{kl}\} : i, k \in [n], j, j', l \in [m], j \neq j', i \neq k\}$. One may notice that finding an r-factorization for $\mathscr{K}_{n\times m}^3$ is equivalent to scheduling the meetings between mathematicians with the above restrictions for the case $\lambda = 1$.

If we replace every edge e of $\mathcal{G}$ by λ copies of e, then we denote the new hypergraph by $\lambda\mathcal{G}$. In this paper, the main result is the following theorem which is obtained by proving a more general result (see Theorem 3.1) using amalgamation–detachment techniques.

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Theorem 1.1. *$\lambda\mathscr{K}^3_{m\times n}$ is $(r_1,\dots,r_k)$-factorable if*

(S1) $3 \mid r_i m$ *for* $i \in [k]$,
(S2) $2 \mid r_i mn$ *for* $i \in [k]$, *and*
(S3) $\sum_{i=1}^k r_i = 3\lambda(n-1)\binom{m}{2}$.

In particular, by letting $r = r_1 = \cdots = r_k$ in Theorem 1.1, we solve the Mathematicians Collaboration Problem in the following case.

Corollary 1.2. *$\lambda\mathscr{K}^3_{m\times n}$ is r-factorable if*

(i) $3 \mid rm$,
(ii) $2 \mid rnm$, *and*
(iii) $r \mid 3\lambda(n-1)\binom{m}{2}$.

The two results above can be seen as analogues of Baranyai's theorem for complete 3-uniform "multipartite" hypergraphs. We note that in fact, Baranyai [9] solved the problem of factorization of complete uniform multipartite hypergraphs, but here we aim to solve this problem under a different notion of "multipartite". In Baranyai's definition, an edge can have at most one vertex from each part, but here we allow an edge to have two vertices from each part (see the definition of $\mathscr{K}^3_{m\times n}$ above). More precise definitions together with preliminaries are given in Section 2, the main result is proved in Section 3, and related open problems are discussed in the last section.

Amalgamation–detachment technique was first introduced by Hilton [10] (who found a new proof for decompositions of complete graphs into Hamiltonian cycles), and was more developed by Hilton and Rodger [11]. Hilton's method was later generalized to arbitrary graphs [15,5], and later to hypergraphs [1,2,4,7] leading to various extensions of Baranyai's theorem (see for example [1,3]). The results of the present paper, mainly relies on those from [1] and [15]. For the sake of completeness, here we give a self contained exposition.

2. More terminology and preliminaries

Recall that an edge can have multiple copies of the same vertex. For the purpose of this paper, all hypergraphs (except when we use the term graph) are 3-uniform, so an edge is always of one of the forms $\{u,u,u\}$, $\{u,u,v\}$, and $\{u,v,w\}$ which we will abbreviate to $\{u^3\}$, $\{u^2,v\}$, and $\{u,v,w\}$, respectively. In a hypergraph $\mathcal{G}$, $\mathrm{mult}_{\mathcal{G}}(.)$ denotes the multiplicity; for example $\mathrm{mult}_{\mathcal{G}}(u^3)$ is the multiplicity of an edge of the form $\{u^3\}$. Similarly, for a graph G, $\mathrm{mult}(u,v)$ is the multiplicity of the edge $\{u,v\}$. A *k-edge-coloring* of a hypergraph $\mathcal{G}$ is a mapping $K : E(\mathcal{G}) \to [k]$, and the sub-hypergraph of $\mathcal{G}$ induced by color i is denoted by $\mathcal{G}(i)$. Whenever it is not ambiguous, we drop the subscripts, and also we abbreviate $d_{\mathcal{G}(i)}(u)$ to $d_i(u)$, $\mathrm{mult}_{\mathcal{G}(i)}(u^3)$ to $\mathrm{mult}_i(u^3)$, etc.

Factorizations of the complete graph, K_n, is studied in a very general form in [12,13], however for the purpose of this paper, a λ-fold version is needed:

Theorem 2.1 (*Bahmanian, Rodger [6, Theorem 2.3]*). *λK_n is $(r_1,\dots,r_k)$-factorable if and only if $r_i n$ is even for $i \in [k]$ and $\sum_{i=1}^k r_i = \lambda(n-1)$.*

Let K_n^* denote the 3-uniform hypergraph with n vertices in which $\mathrm{mult}(u^2,v) = 1$, and $\mathrm{mult}(u^3) = \mathrm{mult}(u,v,w) = 0$ for distinct vertices u,v,w. A (3-uniform) hypergraph $\mathcal{G} = (V,E)$ is *n-partite*, if there exists a partition $\{V_1,\dots,V_n\}$ of V such that for every $e \in E$, $|e \cap V_i| = 1$, $|e \cap V_j| = 2$ for some $i,j \in [n]$ with $i \neq j$. For example, both K_n^* and $\mathscr{K}^3_{m\times n}$ are n-partite. We need another simple but crucial lemma:

Lemma 2.2. *If $r_i n$ is even for $i \in [k]$, and $\sum_{i=1}^k r_i = \lambda(n-1)$, then λK_n^* is $(3r_1,\dots,3r_k)$-factorable.*

Proof. Let $G = \lambda K_n$ with vertex set V. By Theorem 2.1, G is $(r_1,\dots,r_k)$-factorable. Using this factorization, we obtain a k-edge-coloring for G such that $d_{G(i)}(v) = r_i$ for every $v \in V$ and every color $i \in [k]$. Now we form a k-edge-colored hypergraph $\mathcal{H}$ with vertex set V such that $\mathrm{mult}_{\mathcal{H}(i)}(u^2,v) = \mathrm{mult}_{G(i)}(u,v)$ for every pair of distinct vertices $u,v \in V$, and each color $i \in [k]$. It is easy to see that $\mathcal{H} \cong \lambda K_n^*$ and $d_{\mathcal{H}(i)}(v) = 3r_i$ for every $v \in V$ and every color $i \in [k]$. Thus we obtain a $(3r_1,\dots,3r_k)$-factorization for λK_n^*. □

If the multiplicity of a vertex α in an edge e is p, we say that α is *incident* with p distinct *hinges*, say $h_1(\alpha,e),\dots,h_p(\alpha,e)$, and we also say that e is *incident* with $h_1(\alpha,e),\dots,h_p(\alpha,e)$. The set of all hinges in $\mathcal{G}$ incident with α is denoted by $H_{\mathcal{G}}(\alpha)$; so $|H_{\mathcal{G}}(\alpha)|$ is in fact the degree of α.

Intuitively speaking, an *α-detachment* of a hypergraph $\mathcal{G}$ is a hypergraph obtained by splitting a vertex α into one or more vertices and sharing the incident hinges and edges among the subvertices. That is, in an α-detachment $\mathcal{G}'$ of $\mathcal{G}$ in which we split α into α and β, an edge of the form $\{\alpha^p,u_1,\dots,u_z\}$ in $\mathcal{G}$ will be of the form $\{\alpha^{p-i},\beta^i,u_1,\dots,u_z\}$ in $\mathcal{G}'$ for some i, $0 \leqslant i \leqslant p$. Note that a hypergraph and its detachments have the same hinges. Whenever it is not ambiguous, we use d', mult', etc. for degree, multiplicity and other hypergraph parameters in $\mathcal{G}'$.