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Arithmetic into geometric progressions through Riordan arrays

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ABSTRACT

In this paper, by using Riordan arrays and a particular model of lattice paths, we are able to generalize in several ways an identity proposed by Lou Shapiro by giving both an algebraic and a combinatorial proof. The identities studied in this paper allow us to move from an arithmetic progression, and other C-finite sequences, to a geometric progression in terms of Riordan array transformations and vice versa, via the Riordan array inverse.

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1. Introduction

In 1983, Shapiro et al. [21] obtained the following identity

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 5 & 4 & 1 & 0 & 0 & 0 \\ 14 & 14 & 6 & 1 & 0 & 0 \\ 42 & 48 & 27 & 8 & 16 & 0 \\ 132 & 165 & 110 & 44 & 10 & 1 \end{bmatrix} * \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 4^2 \\ 4^3 \\ 4^4 \\ 4^5 \end{bmatrix} \tag{1.1}$$

where the entries in the first column of the matrix are the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ and the entry in row n and column k is determined by summing the elements in row $n-1$ and columns $k-1$ and $k+1$ plus twice the entry in column k. The matrix in (1.1) is often called Shapiro's array due to [19]. A combinatorial proof of the above identity was given by Woan et al. [24] while computing the area of parallelo-polyominoes via generating functions. Later, Chen et al. [2] found out a proof by using explicitly the concept of Riordan arrays and they found many other arrays enjoying a similar property. In particular, they found a matrix identity that extends the sequence $(1, 4, 4^2, 4^3, \ldots)$ to $(1, k, k^2, k^3 \cdots)$ by giving a combinatorial proof in terms of weighted Motzkin paths. A similar generalization is studied from an algebraic point of view in [1].

We briefly recall that a Riordan array is an infinite lower triangular array $(d_{n,k})_{n,k\in\mathbb{N}}$, defined by a pair of formal power series $D = (d(t), h(t))$, such that $d(0) \neq 0$, $h(0) = 0$, $h'(0) \neq 0$ and the generic element $d_{n,k}$ is the nth coefficient in the

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Table 1
A portion of Shapiro's array S and its inverse S^{-1}.

	0	1	2	3	4	5	6
0	1						
1	1	1					
2	2	3	1				
3	5	9	5	1			
4	14	28	20	7	1		
5	42	90	75	35	9	1	
6	132	297	275	154	54	11	1

	0	1	2	3	4	5	6
0	1						
1	−1	1					
2	1	−3	1				
3	−1	6	−5	1			
4	1	−10	15	−7	1		
5	−1	15	−35	28	−9	1	
6	1	−21	70	−84	45	−11	1

series $d(t)h(t)^k$, i.e.:

$$d_{n,k} = [t^n]d(t)h(t)^k, \quad n, k \geq 0.$$

From this definition we have $d_{n,k} = 0$ for $k > n$. These arrays constitute a group introduced in 1991 by Shapiro et al. [20]; the relative literature is vast and constantly growing. Some of their properties and recent applications can be found in [8,10,11]. The product of two Riordan arrays is defined by:

$$D_1 * D_2 = (d_1(t),\ h_1(t)) * (d_2(t),\ h_2(t)) = (d_1(t)d_2(h_1(t)),\ h_2(h_1(t))); \tag{1.2}$$

it corresponds to the usual row-by-column product of two (infinite) matrices. The Riordan array $I = (1,\ t)$ acts as the identity and the inverse of $D = (d(t), h(t))$ is the Riordan array:

$$D^{-1} = (d^*_{n,k}) = (d^*(t), h^*(t)) = \left(\frac{1}{d(\overline{h}(t))}, \overline{h}(t)\right) \tag{1.3}$$

where $\overline{h}(t)$ is the compositional inverse of $h(t)$, hence $h^*(t) = \overline{h}(t)$. An important property of Riordan array concerns the computation of combinatorial sums. In particular we have the following result (see, e.g., [7,14,22]):

$$\sum_{k=0}^{n} d_{n,k}f_k = [t^n]d(t)f(h(t)) \tag{1.4}$$

that is, every combinatorial sum involving a Riordan array can be computed by extracting the coefficient of t^n from the generating function $d(t)f(h(t))$ where $f(t) = \mathcal{G}(f_k) = \sum_{k\geq 0} f_k t^k$ is the generating function of the sequence $(f_k)_{k\in\mathbb{N}}$ and the symbol $\mathcal{G}$ denotes the generating function operator. Due to its importance, relation (1.4) is often called the *fundamental rule* of Riordan arrays. Along the paper, the notation $(f_k)_k$ will be used as an abbreviation of $(f_k)_{k\in\mathbb{N}}$.

The matrix in identity (1.1) corresponds to the Riordan array $R = (C(t)^2, tC(t)^2)$, where $C(t) = (1 - \sqrt{1-4t})/2t$ is the generating function of the Catalan numbers. A slightly different problem posed by Lou Shapiro [18] about twenty years ago goes as follows: consider the Riordan array $S = (C(t),\ tC(t)^2)$ and the sequence O of odd integers; by the properties of Riordan arrays, it is not difficult to show that

$$S * O = (4^n)_n, \quad \text{or} \quad \sum_{k=0}^{n} S_{n,k}(2k+1) = 4^n \tag{1.5}$$

where "$*$" is the usual row-by-column product and the sequences are seen as column vectors, as above. The problem is to find a combinatorial proof of the identity (1.5). In Table 1 we show the upper part of *Shapiro's triangle* S, together with its inverse S^{-1}. It can be used to verify the main properties of these arrays by direct inspection.

After so many years, it is difficult to reconstruct the whole story and several solution were found. Identity (1.5) was also given in [16] and in the literature S is often referred as Radoux's triangle. The problem was recently (in 2011) re-discovered by Gary W. Adamson, as reported in the *On-line Encyclopedia of Integer Sequences* (OEIS) [15] as Sequence A039599; this (double) sequence is nothing but Shapiro's array, and a great number of its properties are listed there. The inverse array is also present in the OEIS as Sequence A085478. More recently, Renzo Sprugnoli re-introduced the problem during the first *Symposium on Riordan arrays and related topics* (Seoul, South Korea, August 2014).

A first motivation to write this paper was to emphasize the connection that can exist between the model and the algebraic approach to a combinatorial problem. Usually, a solution through a model is called a *combinatorial proof*, more appropriate to the nature of the problem and often considered to inspire new and deeper aspects thereof (see, e.g. Stanley [23]). However, algebra simplifies many proofs and suggests extensions difficult to be imagined at the model level; just think of passing from natural numbers (proper of combinatorial objects), to negative integers. The paper starts with an algebraic problem to be solved combinatorially; the solution suggests a generalization of the original problem, which generates curious analogues and many combinatorial sums. A still more general model creates other interesting problems, to be attacked algebraically, and so on in a continuous extensions of problems and results.

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