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# A path Turán problem for infinite graphs

Xing Peng [a], Craig Timmons [b]

[a] Center for Applied Mathematics, Tianjin University, Tianjin 300072, China
[b] Department of Mathematics and Statistics, California State University, Sacramento, United States

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ABSTRACT

Let $G$ be an infinite graph whose vertex set is the set of positive integers, and let $G_n$ be the subgraph of $G$ induced by the vertices $\{1, 2, \ldots, n\}$. An increasing path of length $k$ in $G$, denoted $I_k$, is a sequence of $k+1$ vertices $1 \le i_1 < i_2 < \cdots < i_{k+1}$ such that $i_1, i_2, \ldots, i_{k+1}$ is a path in $G$. For $k \ge 2$, let $p(k)$ be the supremum of $\liminf_{n\to\infty} \frac{e(G_n)}{n^2}$ over all $I_k$-free graphs $G$. In 1962, Czipszer, Erdős, and Hajnal proved that $p(k) = \frac{1}{4}(1 - \frac{1}{k})$ for $k \in \{2, 3\}$. Erdős conjectured that this holds for all $k \ge 4$. This was disproved for certain values of $k$ by Dudek and Rödl who showed that $p(16) > \frac{1}{4}(1 - \frac{1}{16})$ and $p(k) > \frac{1}{4} + \frac{1}{200}$ for all $k \ge 162$. Given that the conjecture of Erdős is true for $k \in \{2, 3\}$ but false for large $k$, it is natural to ask for the smallest value of $k$ for which $p(k) > \frac{1}{4}(1 - \frac{1}{k})$. In particular, the question of whether or not $p(4) = \frac{1}{4}(1 - \frac{1}{4})$ was mentioned by Dudek and Rödl as an open problem. We solve this problem by proving that $p(4) \ge \frac{1}{4}(1 - \frac{1}{4}) + \frac{1}{584064}$ and $p(k) > \frac{1}{4}(1 - \frac{1}{k})$ for $4 \le k \le 15$. We also show that $p(4) \le \frac{1}{4}$ which improves upon the previously best known upper bound on $p(4)$. Therefore, $p(4)$ must lie somewhere between $\frac{3}{16} + \frac{1}{584064}$ and $\frac{1}{4}$.


## 1. Introduction

Turán problems form a cornerstone of extremal graph theory. In general, the Turán problem asks for the maximum number of edges in a graph which does not contain another graph as a subgraph. Turán's Theorem determines this maximum when the forbidden graph is a clique on a fixed number of vertices. Because of its significance, different Turán type problems have been considered in a variety of different settings. One such setting is infinite graphs. Perhaps not surprisingly, Paul Erdős was one of the pioneers of infinite graph theory and we recommend [1] and [6] for excellent discussions of his work in this area as well as many open problems. In this paper, we study a Turán problem on countably infinite graphs that was first considered by Czipszer, Erdős, and Hajnal [2].

Let $G$ be an infinite graph with $V(G) = \{1, 2, 3, \ldots\}$. An *increasing path of length* $k$, denoted by $I_k$, is a sequence of $k+1$ vertices $i_1, \ldots, i_{k+1}$ such that $i_1 < i_2 < \cdots < i_{k+1}$ and $i_j$ is adjacent to $i_{j+1}$ for $1 \le j \le k$. An infinite graph $G$ is $I_k$*-free* if it does not contain an increasing path of length $k$. For an infinite graph $G$, let $G_n$ be the subgraph of $G$ induced by the vertices $\{1, 2, \ldots, n\}$ and $p(G) = \liminf_{n\to\infty} \frac{e(G_n)}{n^2}$. Define the *path Turán number of* $I_k$, denoted $p(k)$, to be the value

$$p(k) = \sup\{p(G) : G \text{ is } I_k\text{-free}\}.$$

Czipszer, Erdős, and Hajnal [2] introduced these path Turán numbers and proved the following.

*E-mail addresses:* x2peng@tju.edu.cn (X. Peng), craig.timmons@csus.edu (C. Timmons).

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**Theorem 1.1** (*Czipszer, Erdős, Hajnal [2]*). *The path Turán numbers $p(2)$ and $p(3)$ satisfy*

$$p(2) = \frac{1}{8} \quad \text{and} \quad p(3) = \frac{1}{6}.$$

They also gave a simple construction that shows

$$p(k) \geq \frac{1}{4}\left(1 - \frac{1}{k}\right) \quad \text{for all } k \geq 2$$

and asked if $p(k) = \frac{1}{4}\left(1 - \frac{1}{k}\right)$ holds for $k \geq 4$. Erdős conjectured in [4] and [5] that $p(k) = \frac{1}{4}\left(1 - \frac{1}{k}\right)$ holds for all $k \geq 2$. In 2008, Dudek and Rödl [3] disproved the conjecture for certain values of $k$ by proving the following result.

**Theorem 1.2** (*Dudek, Rödl [3]*). *The path Turán number $p(16)$ satisfies*

$$p(16) > \frac{1}{4}\left(1 - \frac{1}{16}\right).$$

*Furthermore,*

$$p(k) > \frac{1}{4} + \frac{1}{200}$$

*for all $k \geq 162$.*

The results of [3] and the conjecture $p(k) = \frac{1}{4}\left(1 - \frac{1}{k}\right)$ is mentioned in a survey paper of Komjáth [6] which discusses some of the work of Erdős in infinite graph theory.

Theorems 1.1 and 1.2 suggest the following question: for which values of $k$ does one have

$$p(k) = \frac{1}{4}\left(1 - \frac{1}{k}\right) \tag{1}$$

and in particular, what is the smallest value of $k$ for which (1) holds? Our first result is a construction that shows (1) does not hold for several small values of $k$ and disproves the conjecture of Erdős in the most difficult case; the case when $k = 4$.

**Theorem 1.3.** *If $4 \leq k \leq 15$, then*

$$p(k) > \frac{1}{4}\left(1 - \frac{1}{k}\right).$$

By combining the results of [3] with the results and techniques of this paper, one can show that (1) fails for all $k \geq 4$. For more on this, see Section 5.

Using the argument of [2] we obtained the following upper bound on $p(4)$.

**Theorem 1.4.** *The path Turán number $p(4)$ satisfies*

$$p(4) \leq \frac{1}{4}.$$

In proving Theorem 1.3, we will find a positive constant $c_k$ for which $p(k) \geq \frac{1}{4}(1 - \frac{1}{k}) + c_k$ provided $k \in \{4, 5, \ldots, 15\}$. In particular, we obtain $c_4 = \frac{1}{584064}$ (see Section 3.3) so that by Theorem 1.4,

$$\frac{1}{4}\left(1 - \frac{1}{4}\right) + \frac{1}{584064} \leq p(4) \leq \frac{1}{4}. \tag{2}$$

Determining the exact value of $p(4)$ is a challenging open problem. Probably the lower bound in (2) is closer to the true value of $p(4)$.

The next section introduces a sequence reformulation of the path Turán problem. This reformulation was a key ingredient in the constructions of [3] and we use it in our constructions as well. In Section 3.1 we give our construction method and state our main lemma. Section 3.2 contains the proof of our main lemma. In Section 3.3 we prove Theorem 1.3 and in Section 4 we prove Theorem 1.4.

## 2. Sequences

It will be convenient to work with the sequence formulation of the problem introduced by Dudek and Rödl. Given an $I_k$-free graph $G$ with $V(G) = \mathbb{N}$, partition $\mathbb{N}$ into $k$ sets $N_1, \ldots, N_k$ where

$$N_1(G) = \{n \in \mathbb{N} : \forall m \in \mathbb{N} \text{ with } \{n, m\} \in E(G) \text{ we have } n < m\}$$