Discrete Mathematics 340 (2017) 192–200

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Two-geodesic transitive graphs of valency six[☆]

Wei Jin [a,b], Wei Jun Liu [c,*], Shang Jin Xu [d]

[a] School of Statistics, Jiangxi University of Finance and Economics, Nanchang, Jiangxi, 330013, PR China
[b] Research Center of Applied Statistics, Jiangxi University of Finance and Economics, Nanchang, Jiangxi, 330013, PR China
[c] School of Mathematics and Statistics, Central South University, Changsha, Hunan, 410075, PR China
[d] School of Mathematics and Information Sciences, Guangxi University, Nanning, Guangxi, 530004, PR China

ARTICLE INFO

Article history:
Received 14 February 2015
Received in revised form 18 July 2016
Accepted 3 August 2016

Keywords:
s-geodesic transitive graph
Geodesic transitive graph
Automorphism group

ABSTRACT

For a positive integer s less than or equal to the diameter of a graph Γ, an s-geodesic of Γ is a path $(v_0, v_1, \ldots, v_s)$ such that the distance between v_0 and v_s is s. The graph Γ is said to be s-geodesic transitive, if Γ contains an s-geodesic and its automorphism group is transitive on the set of t-geodesics for all $t \leq s$. In particular, if Γ is s-geodesic transitive with s equal to the diameter of Γ, then Γ is called geodesic transitive. In this paper, we classify the family of finite 2-geodesic transitive graphs of valency 6. Then we completely determine such graphs which are geodesic transitive.

1. Introduction

In this paper, graphs are finite, connected, simple and undirected. In a non-complete graph Γ, a vertex triple (u, v, w) with v adjacent to both u and w is called a 2-*arc* if $u \neq w$, and a 2-*geodesic* if in addition u, w are not adjacent. An *arc* is an ordered pair of adjacent vertices. The graph Γ is said to be 2-*arc transitive* or 2-*geodesic transitive* if its automorphism group $\mathrm{Aut}(\Gamma)$ is transitive on arcs, and also transitive on 2-arcs or 2-geodesics, respectively. Clearly, every 2-geodesic is a 2-arc, but some 2-arcs may not be 2-geodesics. If Γ has girth 3 (length of the shortest cycle is 3), then the 2-arcs contained in 3-cycles are not 2-geodesics. The graph in Fig. 1 is the Kneser graph $KG(6, 2)$ which is 2-geodesic transitive but not 2-arc transitive with valency 6. Thus the family of non-complete 2-arc transitive graphs is properly contained in the family of 2-geodesic transitive graphs.

The first remarkable result about 2-arc transitive graphs comes from Tutte [24,25], and this family of graphs has been studied extensively, see [1,15,18,19,21,27]. The local structure of the family of 2-geodesic transitive graphs was determined in [9]. In [7], the authors classified 2-geodesic transitive graphs of valency 4. Later, in [8], they determined the prime valency 2-geodesic transitive graphs. Hence 6 is the next smallest valency for 2-geodesic transitive graphs to investigate. The first aim of this paper is to give a classification of such graphs.

We denote by $\mathrm{K}_{n[b]}$ the *complete multipartite graph* with n parts of size b where $n \geq 3$, $b \geq 2$, and $\mathrm{K}_{3[2]}$ is the octahedron. Let Ω be a set of cardinality n. Then the *Kneser graph* $KG(n, k)$ is the graph with vertex set all k-subsets of Ω, and two k-subsets are adjacent if and only if they are disjoint. The *triangular graph* $T(n)$ is the graph with vertex set all 2-subsets of Ω, and two 2-subsets are adjacent if and only if they share one common element. Thus $KG(n, 2) = \overline{T(n)}$. (For a graph Γ, its *complement* $\overline{\Gamma}$ is the graph with vertex set $V(\Gamma)$, and two vertices are adjacent if and only if they are not adjacent in Γ.) The *Hamming graph* $\mathrm{H}(d, n)$ has vertex set $\mathbb{Z}_n^d = \mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n$, where $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ is the ring of integers modulo n, and two

☆ Supported by the National Natural Science Foundation of China (Grant No. 11671402, 11661039, 11271208, 11561027 and 2016M590604) and NSF of Jiangxi (20151BAB201001, GJJ150460).

* Corresponding author.
E-mail addresses: jinwei@jxufe.edu.cn (W. Jin), wjliu@csu.edu.cn (W.J. Liu), xusj@gxu.edu.cn (S.J. Xu).

http://dx.doi.org/10.1016/j.disc.2016.08.008
0012-365X/

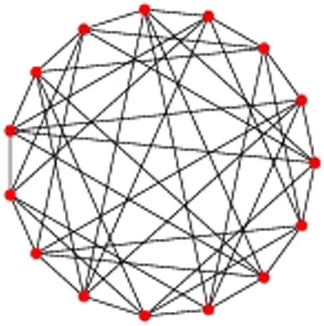

Fig. 1. Kneser graph $KG(6, 2)$.

vertices are adjacent if and only if they have exactly one different coordinate. For $m, n \geq 2$, we define the $(m \times n)$−grid as the graph having vertex set $\{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$, and 2 distinct vertices (i, j) and (r, s) are adjacent if and only if $i = r$ or $j = s$. A subgraph X of Γ is an *induced subgraph* if two vertices of X are adjacent in X if and only if they are adjacent in Γ. For $U \subseteq V(\Gamma)$, we denote by $[U]$ the subgraph of Γ induced by U.

The *diameter* $\text{diam}(\Gamma)$ of a graph Γ is the maximum distance between its two vertices. Let u be a vertex of a graph Γ. We denote by $\Gamma_i(u)$ the set of vertices of Γ at distance i from u, and we write $\Gamma(u) := \Gamma_1(u)$. The sets $\Gamma_i(u)$, for $0 \leq i \leq \text{diam}(\Gamma)$, partition the vertices of Γ. In the characterization of 2-geodesic transitive graphs, the following constants are useful. Our definition is inspired by the concept of intersection array defined for the distance transitive graphs, see [4].

Definition 1. Let Γ be a 2-geodesic transitive graph, $u \in V(\Gamma)$, and let $v \in \Gamma_i(u)$, $i \leq 2$. Then the number of edges from v to $\Gamma_{i-1}(u)$, $\Gamma_i(u)$, and $\Gamma_{i+1}(u)$ is denoted respectively by c_i, a_i and b_i.

The first theorem shows that a 2-geodesic transitive graph of valency 6 with $c_2 \geq 2$ is known.

Theorem 1.1. *Let Γ be a connected 2-geodesic transitive graph of valency 6. Then one of the following holds:*

(1) *Γ is locally connected and is one of the following three graphs: $T(5)$, $K_{3[3]}$ or $K_{4[2]}$.*

(2) *Γ is locally disconnected of girth 3, and either $\Gamma \in \{KG(6, 2), H(2, 4), H(3, 3)\}$ or $c_2 = 1$.*

(3) *Γ has girth at least 4 and is 2-arc transitive. In particular, either $\Gamma \in \{K_{6,6}, \overline{(2\times 7)-\text{grid}}\}$, or $c_2 = 2$ and Γ was classified in [3], or $c_2 = 1$.*

Remark 1.2. Let Γ be a graph in Theorem 1.1(2) such that $c_2 = 1$. Then by [9, Theorem 1.1], the subgraph $[\Gamma(u)] \cong 2K_3$ or $3K_2$. There exists such a Γ. For instance, the generalized hexagon of order $(3, 1)$ has valency 6, $[\Gamma(u)] \cong 2K_3$ and $c_2 = 1$; the halved foster graph has valency 6, $[\Gamma(u)] \cong 3K_2$ and $c_2 = 1$. Further, Theorem 1.4 of [9] showed that there is a bijection between such Γ and the $\mathcal{S}$-point graph of a particular partial linear space $\mathcal{S}$ which has no triangles.

For a positive integer $s \leq \text{diam}(\Gamma)$, an *s-geodesic* of Γ is a path $(v_0, v_1, \ldots, v_s)$ such that the distance between v_0 and v_s is s. The graph Γ is said to be *s-geodesic transitive*, if Γ contains an s-geodesic and $\text{Aut}(\Gamma)$ is transitive on the set of t-geodesics for all $t \leq s$. In particular, if Γ is s-geodesic transitive with $s = \text{diam}(\Gamma)$, then Γ is called *geodesic transitive*. We introduce a weaker symmetry property than geodesic transitivity, namely the distance transitivity. A graph Γ is said to be *distance transitive* if $\text{Aut}(\Gamma)$ is transitive on the ordered pairs of vertices at any given distance. The study of finite distance transitive graphs goes back to Higman's paper [13] in which "groups of maximal diameter" were introduced. These are permutation groups which act distance transitively on some graph. Then distance transitive graphs have been studied extensively and a classification is almost done, see [2,11,14,22,23,26,28]. Note that every geodesic transitive graph is distance transitive.

Our second theorem is to determine all the geodesic transitive graphs of valency 6.

Theorem 1.3. (1) *The Paley graph $P(13)$ and the incidence graph of the 2-$(11, 6, 3)$-design H'_{11} are distance transitive but not geodesic transitive.*

(2) *Let Γ be a connected graph of valency 6 and be not in (1). Then Γ is geodesic transitive if and only if it is distance transitive.*

Remark 1.4. (1) All distance transitive graphs of valency 6 are known, see [11, Lemma 1] and [4, p. 222, 223].

(2) Paley graphs are distance transitive. However, $P(9)$ and $P(5)$ are the only two 2-geodesic transitive graphs, see [16, Theorem 1.2].

(3) The bipartite complement of H'_{11} is the incidence graph of the 2-$(11, 5, 2)$-design. This graph is geodesic transitive of valency 5. (For a bipartite graph Γ with two biparts V_1, V_2, its *bipartite complement* is the bipartite graph with vertex set $V(\Gamma)$ and edge set $\{\{u, v\} \mid u \in V_1, v \in V_2, u, v \text{ are not adjacent in } \Gamma\}$.)