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Note

Super-simple pairwise balanced designs with block sizes 3 and 4

Guangzhou Chen [a,*], Yong Zhang [b], Kejun Chen [c]

[a] Henan Engineering Laboratory for Big Data Statistical Analysis and Optimal Control, School of Mathematics and Information Sciences, Henan Normal University, Xinxiang, 453007, PR China
[b] School of Mathematics and Statistics, Yancheng Teachers University, Yancheng, 224002, PR China
[c] Department of Mathematics, Taizhou University, Taizhou, 225300, PR China

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ABSTRACT

Super-simple designs can be used to provide samples with maximum intersection as small as possible in statistical planning of experiments and can be also applied to cryptography and codes. In this paper, super-simple pairwise balanced designs with block sizes 3 and 4 are investigated and it is proved that the necessary conditions for the existence of a super-simple $(v, \{3, 4\}, \lambda)$-PBD for $2 \le \lambda \le 6$ are sufficient with three possible exceptions.

1. Introduction

A *pairwise balanced design* (or PBD) is a pair $(\mathcal{X}, \mathcal{A})$ such that $\mathcal{X}$ is a set of elements called points, and $\mathcal{A}$ is a set of subsets (called blocks) of $\mathcal{X}$, each of cardinality at least two, such that every pair of points occurs in exactly λ blocks of $\mathcal{A}$. If v is a positive integer and K is a set of positive integers, each of which is greater than one, then we say that $(\mathcal{X}, \mathcal{A})$ is a (v, K, λ)-PBD if $|\mathcal{X}| = v$, and $|A| \in K$ for every $A \in \mathcal{A}$. We denote $B(K, \lambda) = \{v :$ there exists $a\,(v, K, \lambda)$-PBD$\}$. A set K is said to be PBD-*closed* if $B(K, \lambda) = K$.

A PBD is *resolvable* if its blocks can be partitioned into parallel classes; a parallel class is a set of point-disjoint blocks whose union is the set of all points. The notation (v, K, λ)-RPBD is used for a resolvable PBD. When $K = \{k\}$, a (v, K, λ)-PBD is a *balanced incomplete block design*, the notations (v, k, λ)-BIBD and (v, k, λ)-RBIBD are sometimes used in this case.

A design is said to be *simple* if it contains no repeated blocks. A design is said to be *super-simple* if the intersection of any two blocks has at most two elements. When $k = 3$, a super-simple design is just a simple design. When $\lambda = 1$, the designs are necessarily super-simple. A super-simple (v, K_1, λ)-PBD is also a super-simple (v, K_2, λ)-PBD if $K_1 \subseteq K_2$.

The concept of super-simple designs was introduced by Gronau and Mullin [26]. The existence of super-simple designs is an interesting problem by itself, but there are also useful applications. For example, such designs are used in constructing perfect hash families [34] and coverings [7], in the construction of new designs [6] and in the construction of superimposed codes [31]. In statistical planning of experiments, super-simple designs are the ones providing samples with maximum intersection as small as possible. Besides these, super-simple designs have also appeared as sub-orthogonal double covers of certain types of graphs (see, for example [24]). There are other useful applications in [21,29]. Super-simple pairwise balanced designs are powerful for the construction of other types of combinatorial structures, for example super-simple group divisible designs [10].

* Corresponding author.
E-mail address: chenguangzhou0808@163.com (G. Chen).

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Let m denote the smallest integer in K. Define $\alpha(K) = gcd\{k-1 : k \in K\}$ and $\beta(K) = gcd\{k(k-1) : k \in K\}$. The necessary conditions for the existence of a super-simple (v, K, λ)-PBD are $v \geq \lambda(m-2)+2$, $\lambda(v-1) \equiv 0 \pmod{\alpha(K)}$ and $\lambda v(v-1) \equiv 0 \pmod{\beta(K)}$.

The existence of simple $(v, 3, \lambda)$-BIBDs was completely solved by Dehon [22], we state the result below.

Lemma 1.1 ([22]). *There exists a simple $(v, 3, \lambda)$-BIBD if and only if $v > \lambda+2$, $\lambda(v-1) \equiv 0 \pmod 2$ and $\lambda v(v-1) \equiv 0 \pmod 6$.*

The necessary conditions for the existence of a super-simple $(v, 4, \lambda)$-PBD are $v \geq 2\lambda + 2$, $\lambda(v-1) \equiv 0 \pmod 3$ and $\lambda v(v-1) \equiv 0 \pmod{12}$. For the existence of super-simple $(v, 4, \lambda)$-BIBDs, the necessary conditions are known to be sufficient for $\lambda \in \{2-6, 8, 9\}$ (see [4,9,11,12,14,20,23,26,30,35]). Gronau and Mullin [26] solved the case for $\lambda = 2$, and the corrected proof appeared in [30]. The $\lambda = 3$ case was solved by Chen [11]. The $\lambda = 4$ case was solved independently by Adams, Bryant, and Khodkar [4] and Chen [12]. The case of $\lambda = 5$ was solved by Cao, Chen and Wei [9]. The case of $\lambda = 6$ was solved by Chen, Cao and Wei [14]. The case of $\lambda = 8$ was solved by Chen, Sun and Zhang [20]. The case of $\lambda = 9$ was solved by Zhang, Chen and Sun [35]. A survey on super-simple $(v, 4, \lambda)$-BIBDs with $v \leq 32$ appeared in [8]. We summarize these known results in the following result.

Lemma 1.2 ([4,9,11,12,14,20,23,26,30,35]). *The necessary conditions of a super-simple $(v, 4, \lambda)$-BIBD for $\lambda = 2, 3, 4, 5, 6, 8, 9$ are sufficient.*

The necessary conditions for the existence of a super-simple $(v, 5, \lambda)$-BIBD are known to be sufficient for $\lambda \in \{2, 3, 4, 5\}$ (see [1,15–17,23,25,33]). For more results on super-simple design we refer the reader to [2,5,8,13,18,28,32,33] and references therein.

The existence of $(v, \{3, 4\}, 1)$-PBD was stated in [3,27].

Lemma 1.3 ([3,27]). *There exists a $(v, \{3, 4\}, 1)$-PBD if and only if $v \equiv 0, 1 \pmod 3$ and $v \geq 3$.*

In this paper, the existence of a super-simple $(v, \{3, 4\}, \lambda)$-PBD for $2 \leq \lambda \leq 6$ is investigated. The necessary conditions for the existence of such a super-simple design are $v \geq \lambda + 2$ and $\lambda v(v-1) \equiv 0 \pmod 3$. We shall use direct and recursive constructions to show that the necessary conditions are also sufficient with some possible exceptions. Specifically, we shall prove the following theorem.

Theorem 1.4. *The necessary conditions of a super-simple $(v, \{3, 4\}, \lambda)$-PBD for $2 \leq \lambda \leq 6$ are sufficient except possibly for $(v, \lambda) \in \{(18, 5), (30, 5), (42, 5)\}$.*

The paper is organized as follows. Some recursive constructions are provided in Section 2. Some ingredient super-simple designs are given directly by computer programs in Section 3. The proof of our main theorem is given in Section 4. We present one research problem in Section 5.

2. Recursive constructions

In this section, super-simple group divisible designs are used and some recursive constructions using group divisible designs are given, which will be needed in the sequel.

A *group divisible design* (or GDD) is a triple $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ which satisfies the following properties:
(i) $\mathcal{G}$ is a partition of a set $\mathcal{X}$ (of *points*) into subsets called *groups*.
(ii) $\mathcal{B}$ is a set of subsets of $\mathcal{X}$ (called *blocks*) such that a group and a block contain at most one common point.
(iii) Every pair of points from distinct groups occurs in exactly λ blocks.

The *group type* (or *type*) of GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. We usually use an "exponential" notation to describe types: so type $g_1^{u_1} g_2^{u_2} \cdots g_k^{u_k}$ denotes u_i occurrences of g_i, $1 \leq i \leq k$, in the multiset. A GDD with block sizes from a set of positive integers K is called a (K, λ)-GDD. When $\lambda = 1$, we simply write K-GDD. When $K = \{k\}$, we simply write k for K. Taking the groups of a GDD as additional blocks yields a PBD, and taking a parallel class of blocks of a PBD as groups also yields a GDD.

A (k, λ)-GDD of group type v^k is called a *transversal design* and denoted by $TD_\lambda(k, v)$ for short. The following result was stated in [28].

Lemma 2.1 ([28]). *A super-simple $TD_\lambda(4, v)$ exists if and only if $\lambda \leq v$ and (λ, v) is neither $(1, 2)$ nor $(1, 6)$.*

We shall use the following standard recursive constructions in the proof of Theorem 1.4. For details of the following result, we refer the reader to [9,12,15–16], its proof is omitted here.

Lemma 2.2 ([9,12,15–16]). *(Weighting) Let $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ be a super-simple GDD with index λ_1, and let $w : \mathcal{X} \to Z^+ \cup \{0\}$ be a weight function on $\mathcal{X}$, where Z^+ is the set of positive integers. Suppose that for each block $B \in \mathcal{B}$, there exists a super-simple (k, λ_2)-GDD of type $\{w(x) : x \in B\}$, then there exists a super-simple $(k, \lambda_1\lambda_2)$-GDD of type $\{\sum_{x \in G_i} w(x) : G_i \in \mathcal{G}\}$.*

The following result can be regarded as a generalization of the construction of BIBD, see [9,12,15–16].