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Packing large trees of consecutive orders

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ABSTRACT

A conjecture by Bollobás from 1995 (which is a weakening of the famous Tree Packing Conjecture by Gyárfás from 1976) states that any set of k trees $T_n, T_{n-1}, \ldots, T_{n-k+1}$, such that T_{n-i} has $n-i$ vertices, pack into K_n, provided n is sufficiently large. We confirm Bollobás conjecture for trees $T_n, T_{n-1}, \ldots, T_{n-k+1}$, such that T_{n-i} has $k-1-i$ leaves or a pending path of order $k-1-i$. As a consequence we obtain that the conjecture is true for $k \leq 5$.

1. Introduction

A set of (simple) graphs $G_1, G_2, \ldots, G_k$ are said to *pack into a complete graph* K_n (in short pack) if $G_1, G_2, \ldots, G_k$ can be found as pairwise edge-disjoint subgraphs in K_n. Many classical problems in Graph Theory can be stated as packing problems. In particular, H is a subgraph of G if and only if H and the complement of G pack.

A famous tree packing conjecture (TPC) posed by Gyárfás [7] states that any set of n trees $T_n, T_{n-1}, \ldots, T_1$ such that T_i has i vertices pack into K_n. A number of partial results concerning the TPC are known. In particular Gyárfás and Lehel [7] showed that the TPC is true if each tree is either a path or a star. An elegant proof of this result was given by Zaks and Liu [11]. Recently, Joos et al. [9] proved the TPC for all bounded degree trees (earlier, an approximate version of the TPC for all bounded degree trees was proved by Bötcher et al. [4]). In [6] Bollobás suggested the following weakening of TPC:

Conjecture 1. *For every $k \geq 1$ there is an $n_0(k)$ such that if $n > n_0(k)$, then any set of k trees $T_n, T_{n-1}, \ldots, T_{n-k+1}$ such that T_{n-j} has $n-j$ vertices pack into K_n.*

Bourgeois, Hobbs and Kasiraj [3] showed that any three trees T_n, T_{n-1}, T_{n-2} pack into K_n. Balogh and Palmer [2] proved that any set of $k = \frac{1}{10}n^{1/4}$ trees $T_n, \ldots, T_{n-k+1}$ such that no tree is a star and T_{n-j} has $n-j$ vertices pack into K_n. In this paper we confirm the conjecture for new sets of trees.

We say that a tree T has a *pending path* of order t if there exists $e \in E(T)$ such that one component of $T-e$ is a path P of order t and $d_T(v) \leq 2$ for every $v \in V(P)$.

Theorem 2. *Let k be a positive integer and let $n_0(k)$ be a sufficiently large constant depending only on k. If $n > n_0(k)$, then any set of k trees $T_n, T_{n-1}, \ldots, T_{n-k+1}$, such that T_{n-j} has $n-j$ vertices, and T_{n-j} has $k-1-j$ leaves or a pending path of order $k-1-j$, pack into K_n.*

Note that every sufficiently large tree has either at least 4 leaves or a long pending path. Thus, Theorem 2 implies Conjecture 1 for $k \leq 5$.

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Corollary 3. *Let $k \le 5$ be a positive integer and let $n_0(k)$ be a sufficiently large constant depending only on k. If $n > n_0(k)$, then any set of k trees $T_n, T_{n-1}, \dots, T_{n-k+1}$, such that T_{n-j} has $n-j$ vertices pack into K_n.*

Unfortunately, for $k \ge 6$ there are trees with only four leaves and no pending path of order 5 (actually, no pending path of order 2). For example, such trees arise from a path of order $n-2$ by adding one pendant edge to each of the two penultimate vertices of the path). Hence, for $k \ge 6$ Conjecture 1 remains open.

The proofs of preparatory Lemmas 6 and 9 are inspired by Alon and Yuster approach [1], however, many alterations to their method are needed here. These two lemmas are used to first pack one by one most of the trees. The $k-1-j$ vertices of a pending path or a star which are guaranteed to exist by assumption give enough freedom to complete the packing (here we use the mentioned earlier result of Gyárfás and Lehel which proves TPC when every tree is a path or a star).

In what follows we fix an integer $k \ge 1$ and assume that $n \ge n_0(k)$, where $n_0(k)$ is a sufficiently large constant depending only on k.

2. Notation

The notation is standard. In particular $d_G(v)$ (abbreviated to $d(v)$ if no confusion arises) denotes the degree of a vertex v in G, $\delta(G)$ and $\Delta(G)$ denote the minimum and the maximum degree of G, respectively. Furthermore, $N_G(v)$ denotes the set of neighbors of v and, for a subset of vertices $W \subseteq V(G)$,

$$N_G(W) = \bigcup_{w \in W} N_G(w) \setminus W$$

and

$$N_G[W] = N_G(W) \cup W.$$

Let G be a graph and W be any set with $|V(G)| \le |W|$. Given an injection $f : V(G) \to W$, let $f(G)$ denote the graph defined as follows

$$f(G) = (W, \{f(u)f(v) : uv \in E(G)\}).$$

For two graphs G and H let $G \oplus H$ denote the graph defined by

$$G \oplus H = (V(G) \cup V(H), E(G) \cup E(H))$$

(note that $V(G)$ and $V(H)$ do not need to be disjoint).

A packing of k graphs $G_1, \dots, G_k$ with $|V(G_j)| \le n$, $j = 1, \dots, k$, into a complete graph K_n is a set of k injections $f_j : V(G_j) \to V(K_n)$, $j = 1, \dots, k$ such that

$$\text{if } i \ne j \quad \text{then } E(f_i(G_i)) \cap E(f_j(G_j)) = \emptyset.$$

For two graphs G and H with $|V(G)| \le |V(H)|$, we sometimes use an alternative definition. Namely, we call an injection $f : V(G) \to V(H)$ a packing of G and H, if $E(f(G)) \cap E(H) = \emptyset$.

3. Preliminaries

We write $\mathrm{Bin}(n, p)$ for the binomial distribution with n trials and success probability p. Let $X \in \mathrm{Bin}(n, p)$. We will use the following two versions of the Chernoff bound which follows from formulas (2.5) and (2.6) from [8] by taking $t = 2\mu - np$ and $t = np - \mu/2$, respectively.

If $\mu \ge E[X] = np$ then

$$Pr[X \ge 2\mu] \le \exp(-\mu/3). \tag{1}$$

On the other hand, if $\mu \le E[X] = np$ then

$$Pr[X \le \mu/2] \le \exp(-\mu/8). \tag{2}$$

Proposition 4. *Let G be a graph with n vertices and at most m edges. Let $V(G) = \{v_1, \dots, v_n\}$ with $d(v_1) \ge d(v_2) \ge \dots \ge d(v_n)$. Then*

$$d(v_i) \le \frac{2m}{i}.$$

Proof. The proposition is true because

$$2m \ge \sum_{j=1}^{n} d(v_j) \ge \sum_{j=1}^{i} d(v_j) \ge i d(v_i). \quad \square$$