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On the covering number of symmetric groups having degree divisible by six

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ABSTRACT

If a group G is the union of proper subgroups $H_1, \ldots, H_k$, we say that the collection $\{H_1, \ldots, H_k\}$ is a cover of G, and the size of a minimal cover (supposing one exists) is the covering number of G, denoted by $\sigma(G)$. Maróti showed that $\sigma(S_n) = 2^{n-1}$ for n odd and sufficiently large, and he also gave asymptotic bounds for n even. In this paper, we determine the exact value of $\sigma(S_n)$ when n is divisible by six.

1. Introduction

Let G be a group and $\mathcal{A} = \{H_i : 1 \leqslant i \leqslant n\}$ be a collection of proper subgroups of G. If $G = \bigcup_{i=1}^{n} H_i$ (as a set theoretic union), then $\mathcal{A}$ is called a *cover* of G. A cover of size n is said to be *minimal* if no cover of G has fewer than n members. The size of a minimal cover of G, supposing one exists, is called the *covering number* and is denoted by $\sigma(G)$.

The concept of a cover is only well-defined if G is not a cyclic group. Indeed, if G is a cyclic group, then no generator of G is contained in a proper subgroup, and so G has no cover. On the other hand, if G is not cyclic, then one could take all cyclic subgroups as a cover. Moreover, when considering the covering number of a group, it is obvious that the subgroups used in a cover can be restricted to maximal subgroups.

Note that we can consider covers of either finite or infinite groups. Indeed, B.H. Neumann [31] showed that a group is the union of finitely many proper subgroups if and only if it has a finite noncyclic homomorphic image. Thus we can restrict ourselves to finite groups.

The covering number $\sigma(G)$ of a finite group G provides an upper bound for $\omega(G)$, which is defined to be the largest integer m such that there exists a subset S of G of size m with the property that any two distinct elements of S generate G. There has been a great interest in this topic in recent years (see [3–5,22]), especially with regards to the application of $\sigma(G)$ as an upper bound for $\omega(G)$. For surveys regarding the covering number and related problems, see [2,23,33]. For other related work, see [6,8,10,9,13,15–17,19,18,20,26,27,35].

In [11], Cohn conjectures that the covering number of any (noncyclic) solvable group has the form $p^\alpha + 1$, where p is a prime and α is a positive integer. In [34], Tomkinson confirms this conjecture, showing that the covering number of any (noncyclic) solvable group has the form $|H/K| + 1$, where H/K is the smallest chief factor of G having more than one complement in G.

Furthermore, Tomkinson suggests that it might be of interest to determine the covering number of simple groups. Along these lines, the covering number for 2-dimensional linear groups was determined by Bryce, Fedri, and Serena in [7], and the

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covering number for the Suzuki groups $Sz(q)$ was determined by Lucido in [28]. Holmes applied innovative combinatorial and computational techniques using GAP [14] in [21] to calculate the covering number of many sporadic simple groups.

Naturally, there has been significant interest in symmetric and alternating groups. Maróti made substantial progress on both in [30]. For alternating groups, Maróti showed that $\sigma(A_n) \geqslant 2^{n-2}$ for $n \neq 7, 9$ with equality if and only if $n \equiv 2 \pmod 4$ and further proved that $\sigma(A_7) \leqslant 31$ and $\sigma(A_9) \geqslant 80$. Small values of n have been resolved elsewhere. Cohn [11] showed that $\sigma(A_5) = 10$; Kappe and Redden [25] demonstrated that $\sigma(A_7) = 31$, $\sigma(A_8) = 71$, and $127 \leqslant \sigma(A_9) \leqslant 157$; and recently Epstein, Magliveras, and Nikolova-Popova [12] proved that $\sigma(A_9) = 157$ and $\sigma(A_{11}) = 2751$.

For symmetric groups, Maróti showed for n odd that $\sigma(S_n) = 2^{n-1}$ unless $n = 9$ and showed for n even that $\sigma(S_n) \sim \frac{1}{2}\binom{n}{n/2}$. We note that $\sigma(S_4) = 4$ by [34] and $\sigma(S_6) = 13$ by [1]. Kappe, Nikolova-Popova, and the author showed in [24] that $\sigma(S_8) = 64$, $\sigma(S_9) = 256$ (confirming that $\sigma(S_n) = 2^{n-1}$ for all odd n), $\sigma(S_{10}) = 221$, and $\sigma(S_{12}) = 761$, establishing that the upper bound of 761 for $\sigma(S_{12})$ Maróti gave in [30] was in fact the exact value.

It is obvious that computational methods can only be taken so far with symmetric groups of even degree, and the goal of this paper is to analyze a large class of these groups in the same spirit as [30]. We will prove the following theorem, which, when combined with the previous results for S_6 and S_{12}, establishes the covering number for S_n whenever n is divisible by 6:

Theorem 1.1. *Let $n \equiv 0 \pmod 6$, $n \geqslant 24$. If $\sigma(S_n)$ denotes the covering number of S_n, then $\sigma(S_n) = \frac{1}{2}\binom{n}{n/2} + \sum_{i=0}^{n/3-1}\binom{n}{i}$. Moreover, $\sigma(S_{18}) = 36773 = \frac{1}{2}\binom{18}{9} + \sum_{0\leqslant i\leqslant 5, i\neq 2}\binom{18}{i}$. In each of these cases, the minimal cover using only maximal subgroups is unique.*

The following notation will be used throughout the paper. Given an element $g \in S_n$, we say that the permutation g has cycle structure $(n_1, \ldots, n_k)$ with $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$ if g, when written as the product of disjoint cycles, contains cycles of length n_i for $1 \leqslant i \leqslant k$, where $\sum_{i=1}^{k} n_i = n$. For instance, the permutation $(1\ 2)(3\ 4)(5\ 6\ 7\ 8\ 9) \in S_9$ has cycle structure $(2, 2, 5)$, whereas the permutation $(1\ 2)(3\ 4)(5\ 6\ 7\ 8\ 9) \in S_{10}$ has cycle structure $(1, 2, 2, 5)$.

This paper is organized as follows: in Section 2, we provide details about maximal subgroups of symmetric groups; in Section 3, we prove a lemma that provides a sufficient condition for a cover consisting of entire conjugacy classes of maximal subgroups to be minimal; in Section 4, we apply this lemma to the groups S_{18} and S_{24} to establish their covering numbers; and, finally, in Section 5, we apply the lemma to establish the covering number of S_n, where $n \geqslant 30$ and $n \equiv 0 \pmod 6$.

2. Subgroups of symmetric groups

The maximal subgroups of the symmetric group S_n are characterized by the O'Nan–Scott Theorem, which may be stated as follows:

Theorem 2.1 ([32]). *Let H be a maximal subgroup of S_n. Then H is isomorphic to one of the following:*

(i) *$S_k \times S_\ell$, where $k + \ell = n$;*
(ii) *$S_k \operatorname{wr} S_\ell$, where $k\ell = n$;*
(iii) *$S_k \operatorname{wr} S_\ell$, where $k^\ell = n$ and $k > 2$;*
(iv) *$\mathrm{AGL}_d(p)$, where $p^d = n$ for some prime p;*
(v) *$T^k.(\mathrm{Out}(T) \times S_k)$, where T is a nonabelian simple group and $|T|^{k-1} = n$;*
(vi) *an almost simple group.*

For the purposes of this paper, with the exception of singling out the alternating group A_n, there is no need to distinguish among subgroups that fall under (iii)–(vi) of Theorem 2.1. Identifying S_n with its natural action on $\{1, \ldots, n\}$, we will instead divide the maximal subgroups of S_n into the following four classes:

(1) The *alternating group* A_n.
(2) *Intransitive groups*, i.e., those groups isomorphic to $S_k \times S_\ell$, $k+\ell = n$, which stabilize a decomposition of the set $\{1, \ldots, n\}$ into one set of size k and one set of size ℓ.
(3) *Imprimitive groups*, i.e., those groups isomorphic to $S_k \operatorname{wr} S_\ell$, where $k\ell = n$, which stabilize a decomposition of the set $\{1, \ldots, n\}$ into ℓ sets of size k. Note that, unlike the intransitive groups, imprimitive groups are transitive on the set $\{1, \ldots, n\}$.
(4) *Primitive groups*, i.e., those groups that act primitively on $\{1, \ldots, n\}$ and are not the alternating group A_n. These are the groups that are not A_n and fall under (iii)–(vi) of Theorem 2.1.

For large values of n, the primitive groups (that are not A_n) have orders that are very small compared to the orders of the maximal subgroups in classes (1)–(3):

Lemma 2.2 ([29, Corollary 1.2]). *If G is a primitive subgroup of S_n that is not the alternating group A_n and $n > 24$, then $|G| < 2^n$.*