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On the chromatic number of Latin square graphs

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ABSTRACT

The chromatic number of a Latin square is the least number of partial transversals which cover its cells. This is just the chromatic number of its associated Latin square graph. Although Latin square graphs have been widely studied as strongly regular graphs, their chromatic numbers appear to be unexplored. We determine the chromatic number of a circulant Latin square, and find bounds for some other classes of Latin squares. With a computer, we find the chromatic number for all main classes of Latin squares of order at most eight.

1. Introduction and preliminaries

Let $\mathbf{L}$ be a Latin square of order n. The chromatic number of $\mathbf{L}$, denoted by $\chi(\mathbf{L})$, is the minimum number of partial transversals of $\mathbf{L}$ which together cover the cells of $\mathbf{L}$. Since each partial transversal uses at most n of the n^2 cells in $\mathbf{L}$, we observe the following.

Proposition 1. *Every Latin square $\mathbf{L}$ of order n satisfies $\chi(\mathbf{L}) \geq n$, with equality holding if and only if $\mathbf{L}$ has an orthogonal mate.*

Therefore $\chi(\mathbf{L})$ serves as a measure of how close $\mathbf{L}$ is to having an orthogonal mate. We are surprised that reference to this natural invariant seems to be absent from the substantial literature regarding transversals and orthogonality of Latin squares.[1] An early version of some of our results appears in the M.Sc. thesis [12, in Farsi, Persian language] of the fourth author, under the supervision of the third author. In this paper, we find some bounds on $\chi(\mathbf{L})$ for general Latin squares and special classes such as complete Latin squares, Cayley tables of groups, circulants, and all Latin squares of order at most eight.

For the definitions not given here one may refer to [1,4]. Let $\mathbf{L}$ be a Latin square of order n with cells $\{(r, c) \mid r, c \in \{0, 1, 2, \dots, n-1\}\}$; each cell contains a symbol from an alphabet of size n, and no row or column of $\mathbf{L}$ contains a repeated

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1 See the Addendum for further details.

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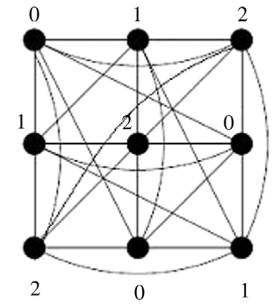

Fig. 1. The graph $\Gamma(L_{\mathbb{Z}_3})$ with each vertex (r, c, s) labeled by s.

symbol. A cell (r, c) containing the symbol $s = \mathbf{L}_{r,c}$ is sometimes represented by the triple (r, c, s). A partial transversal of length k is a set of k cells, where no two cells have the same row, column or symbol. A transversal is a partial transversal of length n. The Latin square $\mathbf{L}$ has an orthogonal mate if and only if it has a decomposition into disjoint transversals. We say that $\mathbf{L}$ is row-complete if every ordered pair of distinct symbols appears (exactly once) in the set

$$\{(s, s') \mid (r, c, s), (r, c+1, s') \in \mathbf{L}, \text{ for } 0 \le r \le n-1 \text{ and } 0 \le c \le n-2\}.$$

The Latin square graph of $\mathbf{L}$ is the simple graph $\Gamma(\mathbf{L})$ whose vertices are the cells of $\mathbf{L}$, and where distinct cells (r, c, s) and (r', c', s') are adjacent if (exactly) one of the equations $r = r', c = c', s = s'$ is satisfied. Accordingly, each edge of $\Gamma(\mathbf{L})$ is called, respectively, a row edge, a column edge or a symbol edge. Latin square graphs were introduced by R.C. Bose [2] as examples of strongly regular graphs; see [8, Section 10.4] for further discussion. Bose used the notation $L_3(n)$ for this graph. But this notation does not specify the Latin square from which the graph arises. So we use the notation $\Gamma(\mathbf{L})$ for the graph corresponding to the given Latin square $\mathbf{L}$. The independent sets of $\Gamma(\mathbf{L})$ are the partial transversals of $\mathbf{L}$, and $\chi(\mathbf{L})$ is the chromatic number of $\Gamma(\mathbf{L})$. The isomorphism class of $\Gamma(\mathbf{L})$ is not affected by relabeling the rows, columns or symbols of $\mathbf{L}$, nor is it changed by applying a fixed permutation to the coordinates of every triple (r, c, s) in $\mathbf{L}$. Thus $\chi(\mathbf{L})$ is an invariant of the main class of $\mathbf{L}$.

Let $(G, \circ)$ be a finite group of order n. A Cayley table for G is an $n \times n$ matrix, denoted L_G, where the cell (i, j) contains the group element $g_i \circ g_j$, for some fixed enumeration $G = \{g_0, \dots, g_{n-1}\}$. It is easy to see that L_G is a Latin square. If G is a cyclic group, then L_G is called a circulant Latin square. Fig. 1 shows the graph of the circulant $L_{\mathbb{Z}_3}$.

We summarize the results of this paper. Let $\mathbf{L}$ be a Latin square of order n.

- $n \le \chi(\mathbf{L}) \le 3n - 2$.
- If $\mathbf{L}$ is row-complete, then $\chi(\mathbf{L}) \le 2n$.
- For large n we have $\chi(\mathbf{L}) = n + o(n)$.
- For every group G of order n, either $\chi(L_G) = n$ or $\chi(L_G) \ge n + 2$.
- If $\mathbf{L}$ is a circulant (that is $\mathbf{L} \cong L_{\mathbb{Z}_n}$), then $\chi(\mathbf{L}) = \begin{cases} n & \text{if n is odd} \\ n+2 & \text{if } n \text{ is even.} \end{cases}$
- If $n \le 8$, then

$$\chi(\mathbf{L}) \le \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n+2 & \text{if } n \text{ is even.} \end{cases} \tag{1}$$

We propose the following.

Conjecture 1. *Every Latin square* $\mathbf{L}$ *satisfies* (1).

Conjecture 1 would surely be challenging to prove, even for Cayley tables of groups. Since $(n+1)(n-1) < n^2$ and $(n+2)(n-2) < n^2$, every Latin square $\mathbf{L}$ which satisfies (1) must have a transversal (if n is odd) or a partial transversal of length $n-1$ (if n is even). So Conjecture 1 would imply two long-standing conjectures of Brualdi–Stein and Ryser.

Conjecture 2 ([5,15]). *Every Latin square of even order n contains a partial transversal of length* $n-1$.

Conjecture 3 ([14]). *Every Latin square of odd order contains a transversal.*

2. Some upper bounds

The graph of a Latin square $\mathbf{L}$ of order n is regular of degree $3n-3$. This immediately gives $\chi(\mathbf{L}) \le 3n-2$. We can improve this bound in case $\mathbf{L}$ has additional structure.

A k-plex is a set of kn cells which has k representatives from each row and each column and each symbol of $\mathbf{L}$. A $(k_1, k_2, \dots, k_d)$-partition is a partition $K_1, K_2, \dots, K_d$ where each K_i is a k_i-plex.