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One-sided interval edge-colorings of bipartite graphs

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ABSTRACT

Let G be a bipartite graph with bipartition (X, Y). An X-interval coloring of G is a proper edge-coloring of G by integers such that the colors on the edges incident to any vertex in X form an interval. Denote by $\chi'_{int}(G, X)$ the minimum k such that G has an X-interval coloring with k colors. In this paper we give various upper and lower bounds on $\chi'_{int}(G, X)$ in terms of the vertex degrees of G. We also determine $\chi'_{int}(G, X)$ exactly for some classes of bipartite graphs G. Furthermore, we present upper bounds on $\chi'_{int}(G, X)$ for classes of bipartite graphs G with maximum degree $\Delta(G)$ at most 9: in particular, if $\Delta(G) = 4$, then $\chi'_{int}(G, X) \leq 6$; if $\Delta(G) = 5$, then $\chi'_{int}(G, X) \leq 15$; if $\Delta(G) = 6$, then $\chi'_{int}(G, X) \leq 33$.

1. Introduction

An *interval coloring* of a graph G is a proper edge-coloring of G by integers such that the colors on the edges incident to any vertex of G form an interval of integers. The notion of interval coloring was introduced by Asratian and Kamalian [4] (available in English as [5]), motivated by the problem of finding compact school timetables, that is, timetables such that the lectures of each teacher and each class are scheduled at consecutive periods. Hansen [11] suggested another scenario (first described by Jesper Bang–Jensen): a school wishes to schedule parent–teacher conferences in time slots so that every person's conferences occur in consecutive slots. A solution exists if and only if the bipartite graph with vertices for the people and edges for the required meetings has an interval coloring.

In the context of edge-colorings, and particularly edge-colorings of bipartite graphs, it is common to consider the general model in which multiple edges are allowed. In this paper, we adopt the convention that "graph" allows multiple edges, and we will explicitly exclude multiple edges when necessary (a *simple graph* is a graph without loops or multiple edges).

All regular bipartite graphs have interval colorings, since they decompose into perfect matchings. Not every graph has an interval coloring, since a graph G with an interval coloring must have a proper $\Delta(G)$-edge-coloring [4]. Furthermore, Sevastjanov [18] proved that determining whether a bipartite graph has an interval coloring is $\mathcal{NP}$-complete; he also gave the first example of a bipartite graph with no interval coloring. (A referee has informed us that the first example of a bipartite graph with no interval coloring was constructed by Mirumyan in 1989, but it was not published; so Sevastjanov's paper [18] contains the first published such example.) Nevertheless, trees [11,5], regular and complete bipartite graphs [11,5], grids [9], and simple outerplanar bipartite graphs [10,6] all have interval colorings.[2]

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[1] Work done while the author was a postdoc at University of Southern Denmark.

[2] A referee has informed us that the result for trees and complete bipartite graphs first appeared in [R.R. Kamlian, Interval colorings of complete bipartite graphs and trees, preprint, Comp. Cen. of Acad. Sci. of Armenian SSR, Yerevan, 1989 (in Russian).]

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A well-known conjecture suggests that all (a, b)-*biregular* graphs have interval colorings (see e.g. [11,14,19]), where a bipartite graph is (a, b)-*biregular* if all vertices in one part have degree a and all vertices in the other part have degree b. By results of [11,13], all $(2, b)$-biregular graphs admit interval colorings (the latter result was also obtained independently by Kostochka [16] and by Kamalian et al.[3]). Several sufficient conditions for a $(3, 4)$-biregular graph G to admit an interval 6-coloring have been obtained [2,17,20]; however, it is still open whether all $(3, 4)$-biregular graphs have interval colorings. In [8] we proved that every $(3, 6)$-biregular graph has an interval 7-coloring and in [7] it was proved that large families of $(3, 5)$-biregular graphs admit interval colorings.

In this paper we study the following relaxation of the problem of finding interval colorings of bipartite graphs: for a bipartite graph G with parts X and Y, an X-*interval coloring* (or *one-sided interval coloring*) of G is a proper edge-coloring of G such that the colors on the edges incident to any vertex of X form an interval of integers. This kind of edge-coloring seems to have been first considered in [5,4]. Note that a one-sided interval coloring of a bipartite graph has a natural interpretation as a timetable where lectures are scheduled at consecutive time slots for the teachers or for the classes. For the graph G in the scenario by Hansen discussed above, a one-sided interval coloring of G corresponds to a schedule where the meetings are consecutive for the parents or for the teachers.

Trivially, every bipartite graph G with parts X and Y has an X-interval coloring with $|E(G)|$ colors. We denote by $\chi'_{int}(G, X)$ the smallest integer t such that there is an X-interval t-coloring of G. Note that, in general, the problem of computing $\chi'_{int}(G, X)$ is $\mathcal{NP}$-hard; this follows from the result of [1] where it is proved that determining whether a given $(3, 6)$-biregular graph has an interval 6-coloring is $\mathcal{NP}$-complete.

Asratian (see e.g. [3]) proved that if a bipartite graph G with parts X and Y satisfies $d_G(x) \geq d_G(y)$ for all edges $xy \in E(G)$, where $x \in X$ and $y \in Y$, and where $d_G(x)$ denotes the degree of x in G, then G has an X-interval coloring such that each vertex $x \in X$ receives colors $1, \dots, d_G(x)$ on its incident edges. Moreover, in [5,4] it is proved that for any t satisfying $\chi'_{int}(G, X) \leq t \leq |E(G)|$, there is an X-interval t-coloring of G such that for $i \in \{1, \dots, t\}$, some edge of G is colored i.

For a bipartite graph G with parts X and Y, denote by $\delta(X)$ the minimum degree in X, by $\Delta(X)$ the maximum degree in X, and by $\Delta(G)$ the maximum degree of G. In this paper we obtain a number of results on one-sided interval colorings of bipartite graphs; in particular we prove the following:

- If M is a maximum matching in G, then
$$\left\lceil \frac{|X|}{|M|} \right\rceil \delta(X) \leq \chi'_{int}(G, X) \leq \left\lceil \frac{|X|}{\delta(X)} \right\rceil \Delta(X).$$
A slightly weaker form of the second inequality was first obtained by Kamalian [15].
- If $\Delta(G) = D$ and $\delta(X) = D - 1$, then $\chi'_{int}(G, X) \leq 2D - 2$; if G is $(D - 1, D)$-biregular with $\Delta(X) = D - 1$, then $\chi'_{int}(G, X) = 2D - 2$.
- If $\Delta(G) = D$ and the vertex degrees in X are in $\{1, d_1, \dots, d_k, D - 1, D\}$, where
$$1 < d_1 < \cdots < d_k < D - 1,$$
then
$$\chi'_{int}(G, X) \leq 2D + d_k - 3 + \sum_{i=1}^{k} \left(\binom{D}{d_i} - (D + d_i - 1) \right) d_i.$$
- If $\Delta(X) = 3$ and $\Delta(Y) = 5$, then $\chi'_{int}(G, X) \leq 7$; if $\Delta(X) = 3$ and $\Delta(Y) = 6$, then $\chi'_{int}(G, X) \leq 13$.
- If $d_G(x) = 3$ for each $x \in X$ and $\Delta(Y) \leq 9$, then $\chi'_{int}(G, X) \leq 17$; if $d_G(x) = 4$ for each $x \in X$ and $\Delta(Y) \leq 8$, then $\chi'_{int}(G, X) \leq 10$.
- If $\Delta(G) = 4$, then $\chi'_{int}(G, X) \leq 6$; if $\Delta(G) = 5$, then $\chi'_{int}(G, X) \leq 15$; if $\Delta(G) = 6$, then $\chi'_{int}(G, X) \leq 33$.

All our proofs of upper bounds on $\chi'_{int}(G, X)$ are constructive and yield polynomial algorithms for constructing the corresponding colorings.

2. General results

We first introduce some terminology and notation and also state some preliminary results. Throughout the paper, we let an X, Y-*bigraph* be a bipartite graph with bipartition (X, Y), and we use $E(G)$ for the edge set of a graph G. When considering vertices x and y in a X, Y-bigraph, we shall always assume that $x \in X$ and $y \in Y$, unless otherwise stated. Moreover, we use the convention that if an X, Y-bigraph G is (a, b)-biregular, then the vertices in X have degree a.

If $S \subseteq V(G)$, then $G[S]$ denotes the subgraph of G induced by S; and if $E' \subseteq E(G)$, then $G[E']$ is the subgraph of G induced by E', i.e., the subgraph induced by all vertices which are endpoints of edges in E'.

[3] A referee has informed us that the fact that $(2, b)$-biregular graphs have interval colorings was also obtained by Kamalian and Mirumyan in [Kamalian, Mirumyan, Interval edge-colorings of bipartite graphs of some class, Dokl. NAN RA, 97 (1997), pp. 3-5 (in Russian)].