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Pentavalent symmetric graphs of order twice a prime power

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ABSTRACT

A connected symmetric graph of prime valency is *basic* if its automorphism group contains no nontrivial normal subgroup having more than two orbits. Let p be a prime and n a positive integer. In this paper, we investigate properties of connected pentavalent symmetric graphs of order $2p^n$, and it is shown that a connected pentavalent symmetric graph of order $2p^n$ is basic if and only if it is either a graph of order 6, 16, 250, or a graph of three infinite families of Cayley graphs on generalized dihedral groups—one family has order $2p$ with $p = 5$ or $5 \mid (p-1)$, one family has order $2p^2$ with $5 \mid (p \pm 1)$, and the other family has order $2p^4$. Furthermore, the automorphism groups of these basic graphs are computed. Similar works on cubic and tetravalent symmetric graphs of order $2p^n$ have been done.

It is shown that basic graphs of connected pentavalent symmetric graphs of order $2p^n$ are symmetric elementary abelian covers of the dipole Dip_5, and with covering techniques, uniqueness and automorphism groups of these basic graphs are determined. Moreover, symmetric $\mathbb{Z}_p^n$-covers of the dipole Dip_5 are classified. As a byproduct, connected pentavalent symmetric graphs of order $2p^2$ are classified.

1. Introduction

Let G be a permutation group on a set Ω and $\alpha \in \Omega$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. We say that G is *semiregular* on Ω if $G_\alpha = 1$ for every $\alpha \in \Omega$ and *regular* if G is transitive and semiregular. We will use the symbol $\mathbb{Z}_n$, both for the cyclic group of order n and for the ring of integers modulo n (and for the field of order n if n is a prime). Denote by $\mathbb{Z}_n^*$ the multiplicative group of units of $\mathbb{Z}_n$, by D_n the dihedral group of order $2n$, and by A_n and S_n the alternating group and the symmetric group of degree n, respectively.

All graphs in this article are finite, connected and simple, unless explicitly stated. For a graph Γ, let $V(\Gamma)$, $E(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ denote the vertex set, edge set and full automorphism group of Γ, respectively. An *s-arc* in a graph Γ is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of $s+1$ vertices such that $\{v_{i-1}, v_i\} \in E(\Gamma)$ for $1 \le i \le s$ and $v_{i-1} \ne v_{i+1}$ for $1 \le i \le s-1$, and a 1-arc is also called an *arc*. For a subgroup G of $\mathrm{Aut}(\Gamma)$ of a graph Γ, the graph Γ is said to be (G, s)-*arc-transitive* or (G, s)-*regular* if G acts transitively or regularly on the set of s-arcs of Γ, and (G, s)-*transitive* if G acts transitively on the set of s-arcs but not on the set of $(s+1)$-arcs of Γ. A graph Γ is said to be *s-arc-transitive*, *s-regular* or *s-transitive* if it is $(\mathrm{Aut}(\Gamma), s)$-arc-transitive, $(\mathrm{Aut}(\Gamma), s)$-regular or $(\mathrm{Aut}(\Gamma), s)$-transitive. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive* or *symmetric*.

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Let Γ be a graph and $N \leq \mathrm{Aut}(\Gamma)$. The *quotient graph* Γ_N of Γ relative to N is defined as the graph with vertices the orbits of N on $V(\Gamma)$ and with two orbits adjacent if there is an edge in Γ between those two orbits. The theory of quotient graph is widely used to investigate symmetric graphs. Let Γ be a symmetric graph and $N \lhd \mathrm{Aut}(\Gamma)$. If Γ and Γ_N have same valency, the graph Γ is said to be a *normal cover* of Γ_N and the graph Γ_N is said to be a *normal quotient* of Γ. In this case, N is semiregular on $V(\Gamma)$. There are two steps to study a symmetric graph Γ—the first step is to investigate normal quotient graph Γ_N for some normal subgroup N of $\mathrm{Aut}(\Gamma)$ and the second step is to reconstruct the original graph Γ from the normal quotient Γ_N by using covering techniques. This is usually done by taking the normal subgroup N as large as possible and then the graph Γ is reduced to a 'basic graph'. The situation seems to be somewhat more promising with 2-arc-transitive graphs, and the strategy for the structural analysis of these graphs, based on taking normal quotients, was first laid out by Praeger (see [46–48]). The strategy works for locally primitive graphs, that is, vertex-transitive graphs with vertex stabilizers acting primitively on the corresponding neighbours sets (see [45,49]).

As for the first step, let us define some notations. A graph Γ is called *basic* if Γ has no proper normal quotient. Then a locally primitive graph is basic if and only if it has no nontrivial normal subgroup having more than two orbits. A graph is *quasiprimitive* if every nontrivial normal subgroup of its automorphism group is transitive, and is *biquasiprimitive* if it has a nontrivial normal subgroup with two orbits but no such subgroup with more than two orbits. Therefore for locally primitive graphs, basic graphs are equivalent to quasiprimitive or biquasiprimitive graphs, which have received most of the attention thus far. In [27], Ivanov and Praeger completed the classification of quasiprimitive 2-arc-transitive graphs of affine type, and Baddeley gave a detailed description of quasiprimitive 2-arc-transitive graphs of twisted wreath type [2]. A similar description of 2-arc-transitive graphs associated with Suzuki groups and Ree groups was obtained by Fang and Praeger [18,19]. Classifications of quasiprimitive 2-arc-transitive graphs of odd order and prime power order have been completed by Li [29–31], and based on this approach, finite vertex-primitive 2-arc-regular graphs have been classified [17] and finite 2-arc-transitive Cayley graphs of abelian groups have been determined [33]. Most recently, symmetric graphs of diameter 2 admitting an affine-type quasiprimitive group were investigated by Amarra et al. [1], and an infinite family of biquasiprimitive 2-arc-transitive cubic graphs was constructed by Devillers et al. [7].

Based on the stabilizers of pentavalent symmetric graphs given by Guo and Feng [23], in this paper we prove that normal quotient graphs of connected pentavalent symmetric graph of order twice a prime power can be K_6, FQ_4 (the folded hypercube of order 16), $\mathcal{CD}_p$ ($p = 5$ or $5 \mid (p-1)$), $\mathcal{CGD}^1_{p^2}$ ($p = 5$ or $5 \mid (p-1)$), $\mathcal{CGD}^2_{p^2}$ ($5 \mid (p \pm 1)$), $\mathcal{CGD}_{p^3}$ ($p = 5$ or $5 \mid (p-1)$), or $\mathcal{CGD}_{p^4}$, where the graphs are defined in Eqs. (1)–(6). Automorphism groups of these normal quotients are computed, and among them, basic ones are determined, which are K_6, FQ_4, $\mathcal{CGD}_{5^3}$, $\mathcal{CD}_p$ ($p = 5$ or $5 \mid (p-1)$), $\mathcal{CGD}^2_{p^2}$ ($5 \mid (p \pm 1)$) and $\mathcal{CGD}_{p^4}$ ($p = 3$ or $p \geq 7$). Similar works on cubic symmetric graphs and tetravalent 2-arc-transitive graphs of order twice a prime power were done by Kwak and the first two authors [13,56].

As for the second step, regular covering (for notation, see Section 4) is becoming an active topic in algebraic graph theory. In [10], regular covers of complete graphs whose group of covering transformations is either cyclic or isomorphic to $\mathbb{Z}_p^2$, p a prime, and whose fibre-preserving subgroup of automorphisms acts 2-arc-transitively, were classified. This result has been extended to the case where the group of covering transformations is isomorphic to $\mathbb{Z}_p^3$, p a prime [9]. Some general methods of elementary abelian coverings were developed in [8,36,37]. By using the method developed in [36], Malnič and Potočnik [39] classified all vertex-transitive elementary abelian covers of the Petersen graph. Symmetric cyclic or elementary abelian covers of the complete graph K_4, the complete bipartite graph $K_{3,3}$, the cube Q_3 and the Petersen graph O_3, were classified in [11,12,14,15,20]. Symmetric elementary abelian covers of the unique connected cubic symmetric graph of order 14, 16 or 18 were classified in [42–44]. By using the above covers, together with group theory techniques, many classifications of symmetric graphs have been obtained—for example, symmetric cubic graphs of order rp or tp^2 were classified for each $2 \leq r \leq 16$ and $2 \leq t \leq 10$. Classification of symmetric graphs with a given order has been widely investigated, and for more results, see [4,5,32,50,51,53]. In the above papers, graphs and their covers are simple, that is, no loops and multiple edges. Regular covers of non-simple graphs were also considered in literature and in this case, automorphism groups of non-simple graphs are usually considered as permutation groups on the sets of arcs of these graphs. For example, to classify tetravalent non-Cayley graph of order four times a prime, Zhou [55] considered vertex-transitive covers of non-simple graphs of order 4.

To determine the uniqueness of normal quotient graphs of connected pentavalent symmetric graph of order twice a prime power for some given orders and to compute their automorphism groups, covering techniques are employed. In this paper we first prove that these normal quotients are symmetric elementary abelian covers of the dipole Dip_5 and then determine all symmetric elementary abelian covers of Dip_5, which consist of four infinite families of Cayley graphs on generalized dihedral groups, that is, the graphs $\mathcal{CGD}^1_{p^2}$ ($p = 5$ or $5 \mid (p-1)$), $\mathcal{CGD}^2_{p^2}$ ($5 \mid (p \pm 1)$), $\mathcal{CGD}_{p^3}$ ($p = 5$ or $5 \mid (p-1)$) and $\mathcal{CGD}_{p^4}$. These covers are not isomorphic to each other and their full automorphism groups are computed. As an application, pentavalent symmetric graphs of order twice a prime square are classified.

2. Preliminaries

In this section, we describe some preliminary results which will be used later. First we describe stabilizers of connected pentavalent symmetric graphs.