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Enumeration of generalized lattice paths by string types, peaks, and ascents

Youngja Park, SeungKyung Park *

Department of Mathematics, Yonsei University, 50 Yonsei Ro, Seoul 120-749, Republic of Korea

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ABSTRACT

We consider lattice paths with arbitrary step sizes, called generalized lattice paths, and we enumerate them with respect to string types of $\mathbf{d}^p\mathbf{u}^q\mathbf{d}^r$ for any positive integers p, q, and r. We find that both numbers of types $\mathbf{d}^p\mathbf{u}\mathbf{d}^r$ and $\mathbf{d}^p\mathbf{u}^{2+}\mathbf{d}^r$ are independent of the number of i flaws for $1 \le i \le n-1$, i.e., they satisfy the Chung–Feller property, where $\mathbf{u}$ is a unit step, $\mathbf{u}^k$ is an up *step* of length k, and $\mathbf{u}^{2+} = \mathbf{u}^{s_1}\mathbf{u}^{s_2}\cdots\mathbf{u}^{s_t}$ with $\sum_{i=1}^{t} s_i \ge 2$. The enumeration of generalized lattice paths by peaks and by ascents is also studied.

1. Introduction

Lattice paths have been studied by many mathematicians and produced numerous interesting results (see a wonderful survey by Humphreys [8]).

Some have been generalizing lattice paths by considering various step sizes. By a *generalized n-lattice path* we mean a lattice path from (0, 0) to (2n, 0) with step set $\{\mathbf{u}^k = (k, k), \mathbf{d}^m = (m, -m) \mid k, m \in \mathbb{Z}^+\}$. If it does not go below the x-axis, we call it a *generalized n-Dyck path*. Sulanke [18] and Woan [20] considered generalized n-Dyck paths among other things. It was Coker [4] who investigated generalized n-Dyck paths thoroughly. Yeh et al. [6,10] considered generalized lattice paths in terms of concatenation of vectors and found some interesting properties of them. Rukavicka [16] combinatorially proved a formula for counting Dyck paths with a finite number of horizontal steps of any size whose sum is the terminal point of the x-axis. Huq [9] also studied lattice paths with steps (1, 1) and $(1, -r)$ for any positive integer r and found a generalized Chung–Feller theorem on the paths. Recently, Huh and Park [7] investigated the generalized n-lattice paths and found a generalized Chung–Feller property for the paths with arbitrary step sizes.

The Chung–Feller theorem was first proved by MacMahon [13] and has been reproved by several mathematicians by different methods (for example, Chung and Feller [3], Narayana [15], and Chen [1]). It states that the number of n-Dyck paths with unit steps and with k up steps below the x-axis is independent of k. This simple but beautiful theorem seems to be the one that has given significant impacts on the enumeration of lattice paths. Naturally, it also has been tried to be generalized as well, because it is automatically related to any form of generalized lattice paths. It is also called a Chung–Feller property or a uniform partition property. There have been many interesting results on the property itself and on the generalized forms. For detailed information see [11,6,9,10,7].

* Corresponding author.
E-mail addresses: ojpark@yonsei.ac.kr (Y. Park), sparky@yonsei.ac.kr (S. Park).

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A closed form for counting the number of elements of a set of generalized lattice paths with certain restrictions can be obtained by finding a Chung–Feller property on the set, if there exists. The results in this paper use refinements of a Chung–Feller property.

Enumeration on lattice paths with various combinatorial aspects of strings with unit steps has been studied extensively. There are too many papers to mention, so we just refer to [5,19] for the enumeration of Dyck paths on several types of strings, and to [12,14] for lattice paths on all string types of length 2 or 3. Especially, for recent and detailed reference see [14].

In this paper, we count generalized n-lattice paths with respect to certain string types, peaks, and ascents by purely combinatorial ways.

Before we state what we will do in detail, we would like to introduce the following notations first. Please notice that we differently use the terms of *steps* and *units*.

- $\mathbf{u}^k$ ($\mathbf{d}^k$): an up (a down) *step* of length $k \in \mathbb{Z}^+$, respectively.
- u (d): an up (a down) *unit*, respectively. Thus $\mathbf{u}^k$ is an up step of k up units u. If $k = 1$, $\mathbf{u}$ and $\mathbf{d}$ become the unit steps in the generalized lattice paths. However, if we need to talk about ordinary lattice paths with unit steps, we will use u and d for up and down steps.
- $\mathbf{d}^p\mathbf{u}^q\mathbf{d}^r$ represents a string of a down step of length p, an up step of length q, and a down step of length r, where p, q, and r are positive integers. We will consider the cases of $\mathbf{d}^p\mathbf{u}\mathbf{d}^r$ and $\mathbf{d}^p\mathbf{u}^{2+}\mathbf{d}^r$, where $\mathbf{d}^p\mathbf{u}^{2+}\mathbf{d}^r$ is the set of strings $\mathbf{d}^p\mathbf{u}^{s_1}\mathbf{u}^{s_2}\cdots\mathbf{u}^{s_t}\mathbf{d}^r$, with $p, r \geq 1$ and $\sum_{i=1}^{t} s_i \geq 2$.
- Every up unit below the x-axis in a generalized lattice path is called *flaw*.
- $\mathbf{D}_j(n, i)$ denotes the set of all generalized lattice paths with i flaws and j $\mathbf{d}^p\mathbf{u}\mathbf{d}^r$'s.
- $\mathbf{D}_k^{2+}(n, i)$ denotes the set of all generalized lattice paths with k $\mathbf{d}^p\mathbf{u}^{2+}\mathbf{d}^r$ strings with i flaws.
- An *ascent* is a maximal segment of consecutive up steps.

It seems that the number of generalized lattice paths containing strings of the form $\mathbf{d}^p\mathbf{u}^q\mathbf{d}^r$ for any positive integers p, q, and r does not give a nice and simple formula. However, if we separate the strings into two cases as $\mathbf{d}^p\mathbf{u}\mathbf{d}^r$ and $\mathbf{d}^p\mathbf{u}^{2+}\mathbf{d}^q$, then we have closed formulas because they are independent of the number of flaws from 1 to $n-1$ (the Chung–Feller property). For the unit steps the number of lattice paths with j dud's and with i flaws was enumerated by the Motzkin numbers as in [19,14], where they dealt with udu strings which is the same as dud by symmetry. However, the number $|\mathbf{D}_j(n, i)|$ involves the Catalan number, which is a bit surprising. The other number $|\mathbf{D}_k^{2+}(n, i)|$ is derived by a combinatorial argument, whose special case of $i = 0$, $|\mathbf{D}_k^{2+}(n, 0)|$, gives another combinatorial interpretation for the formula mentioned in [4,2].

Topics of these kinds will be discussed in Section 2. In the last section, we count the number of generalized lattice paths by peaks and by ascents. Even if the former does not satisfy the Chung–Feller property, we find a closed form by combinatorial methods. On the other hand, the latter satisfies Chung–Feller property, which gives us a fairly simple formula.

2. Generalized lattice paths containing $\mathbf{d}^p\mathbf{u}\mathbf{d}^r$ and $\mathbf{d}^p\mathbf{u}^{2+}\mathbf{d}^r$ strings

In this section we will enumerate the sets $\mathbf{D}_j(n, i)$ and $\mathbf{D}_k^{2+}(n, i)$. We try to find a closed formula for the number $|\mathbf{D}_j(n, i)|$ by counting $|\mathbf{D}_j(n, 1)|$ combinatorially, after proving the Chung–Feller property for those paths with $1 \leq i \leq n-1$ flaws. To do this we will first construct a bijection between the set $\mathbf{D}_j(n, i)$ and the set $\mathbf{D}_j(n, i+1)$ for $1 \leq i \leq n-2$, which is based on a variation of the bijection ψ given for the ordinary unit steps in [14].

We use the following notations for factorization.

- P_1 and P_2 are generalized lattice paths that do not go below the x-axis, i.e., they are generalized Dyck paths.
- N_1 and N_2 are generalized lattice paths that do not go above the x-axis.
- Q is a generalized lattice path that starts and ends with d.
- If $\mathtt{u}$ is the last up unit starting from the x-axis and $\mathtt{d}$ is the first down unit ending at the x-axis after the $\mathtt{u}$, then we call such d as the *matching* $\mathtt{d}$ for the $\mathtt{u}$. Similarly, if $\mathtt{u}$ is the first up unit ending at the x-axis, then the *matching* $\mathtt{d}$ is the first down unit starting from the x-axis before the $\mathtt{u}$.
- Unless $\mathtt{u} = \mathbf{u}$ ($\mathtt{d} = \mathbf{d}$), the unit up (down) step in the generalized lattice paths, $\mathtt{u}$ ($\mathtt{d}$) becomes a part of an up (a down) step, respectively.

Consider a path P in $\mathbf{D}_j(n, i)$ that is factorized as $P = P_1QN_1\mathtt{u}P_2\mathtt{d}N_2$. Such $\mathtt{u}$ and $\mathtt{d}$ exist since $1 \leq i \leq n-2$. We construct a bijection ψ from $\mathbf{D}_j(n, i)$ to $\mathbf{D}_j(n, i+1)$ that preserves the number of $\mathbf{d}^p\mathbf{u}\mathbf{d}^r$'s while increases the number of flaws by one.

A vertex in a generalized lattice path is either an "endvertex" that is an end and a start vertex so that it separates steps, or a "midvertex" that is in the middle of a step. We denote (AB) as a vertex between two nonempty paths A and B. The *parity* of the vertex (AB) refers to (AB) being an endvertex or a midvertex. Thus the parity of two vertices (AB) and (CD) is the same, denoted by $(AB) \leftrightarrow (CD)$, means that both of them are either endvertices or midvertices. The case of $(AB) \leftrightarrow (AB)$ represents that the parity of the same vertex is preserved, so we will not say any on this natural preservation case.

To define ψ we have two cases to consider:

(1) Case of $Q \neq \emptyset$:
$P = P_1QN_1\mathtt{u}P_2\mathtt{d}N_2 \leftrightarrow \psi(P) = P_1\mathtt{d}N_1\mathtt{u}P_2QN_2$ with
$(P_1Q) \leftrightarrow (P_1\mathtt{d})$, $(QN_1) \leftrightarrow (\mathtt{d}N_1)$, $(P_2\mathtt{d}) \leftrightarrow (P_2Q)$, and $(\mathtt{d}N_2) \leftrightarrow (QN_2)$.