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The chromatic spectrum of signed graphs

Yingli Kang [1], Eckhard Steffen *

Paderborn Center for Advanced Studies and Institute for Mathematics, Paderborn University, Warburger Str. 100, 33102 Paderborn, Germany

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ABSTRACT

The chromatic number $\chi((G,\sigma))$ of a signed graph (G,σ) is the smallest number k for which there is a function $c : V(G) \to \mathbb{Z}_k$ such that $c(v) \neq \sigma(e)c(w)$ for every edge $e = vw$. Let $\Sigma(G)$ be the set of all signatures of G. We study the chromatic spectrum $\Sigma_\chi(G) = \{\chi((G,\sigma)) \colon \sigma \in \Sigma(G)\}$ of (G,σ). Let $M_\chi(G) = \max\{\chi((G,\sigma)) \colon \sigma \in \Sigma(G)\}$, and $m_\chi(G) = \min\{\chi((G,\sigma)) \colon \sigma \in \Sigma(G)\}$. We show that $\Sigma_\chi(G) = \{k \colon m_\chi(G) \leq k \leq M_\chi(G)\}$. We also prove some basic facts for critical graphs.

Analogous results are obtained for a notion of vertex-coloring of signed graphs which was introduced by Máčajová, Raspaud, and Škoviera in Máčajová et al. (2016).

1. Introduction

Graphs in this paper are simple and finite. The vertex set of a graph G is denoted by $V(G)$, and the edge set by $E(G)$. A signed graph (G,σ) is a graph G and a function $\sigma : E(G) \to \{\pm 1\}$, which is called a signature of G. The set $N_\sigma = \{e : \sigma(e) = -1\}$ is the set of negative edges of (G,σ) and $E(G) - N_\sigma$ the set of positive edges. For $v \in V(G)$, let $E(v)$ be the set of edges which are incident to v. A switching at v defines a graph (G,σ') with $\sigma'(e) = -\sigma(e)$ for $e \in E(v)$ and $\sigma'(e) = \sigma(e)$ otherwise. Two signed graphs (G,σ) and (G,σ^*) are equivalent if they can be obtained from each other by a sequence of switchings. We also say that σ and σ^* are equivalent signatures of G.

A circuit in (G,σ) is balanced, if it contains an even number of negative edges; otherwise it is unbalanced. The graph (G,σ) is unbalanced, if it contains an unbalanced circuit; otherwise (G,σ) is balanced. Moreover, (G,σ) is antibalanced, if every circuit contains an even number of positive edges. It is well known (see e.g. [3]) that (G,σ) is balanced if and only if it is equivalent to the signed graph with no negative edges, and (G,σ) is antibalanced if it is equivalent to the signed graph with no positive edges. Note, that a balanced bipartite graph is also antibalanced. The underlying unsigned graph of (G,σ) is denoted by G.

In the 1980s Zaslavsky [6,5,7] started studying vertex colorings of signed graphs. The natural constraints for a coloring c of a signed graph (G,σ) are, that (1) $c(v) \neq \sigma(e)c(w)$ for each edge $e = vw$, and (2) that the colors can be inverted under switching, i.e., equivalent signed graphs have the same chromatic number. In order to guarantee these properties of a coloring, Zaslavsky [6] used the set $\{-k, \dots, 0, \dots, k\}$ of $2k+1$ "signed colors" and studied the interplay between colorings and zero-free colorings through the chromatic polynomial.

Recently, Máčajová, Raspaud, and Škoviera [2] modified this approach. If $n = 2k+1$, then let $M_n = \{0, \pm 1, \dots, \pm k\}$, and if $n = 2k$, then let $M_n = \{\pm 1, \dots, \pm k\}$. A mapping c from $V(G)$ to M_n is a signed n-coloring of (G,σ), if $c(v) \neq \sigma(e)c(w)$

* Corresponding author.
E-mail addresses: yingli@mail.upb.de (Y. Kang), es@upb.de (E. Steffen).

[1] Fellow of the International Graduate School "Dynamic Intelligent Systems".

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for each edge $e = vw$. They define $\chi_{\pm}((G, \sigma))$ to be the smallest number n such that (G, σ) has a signed n-coloring. We also say that (G, σ) is signed n-chromatic.

In [1] we study circular coloring of signed graphs. The related integer k-coloring of a signed graph (G, σ) is defined as follows. Let $\mathbb{Z}_k$ denote the cyclic group of integers modulo k, and the negative of an element x is denoted by $-x$. A function $c : V(G) \to \mathbb{Z}_k$ is a k-coloring of (G, σ), if $c(v) \neq \sigma(e)c(w)$ for each edge $e = vw$. Clearly, such colorings satisfy the constraints (1) and (2) of a vertex coloring of signed graphs. The chromatic number of a signed graph (G, σ) is the smallest k such that (G, σ) has a k-coloring. We also say that (G, σ) is k-chromatic.

The following proposition describes the relation between these two coloring parameters for signed graphs. It is already proved in [1]. We add the short proof for the sake of self-containment.

Proposition 1.1 (*[1]*). *If (G, σ) is a signed graph, then $\chi_{\pm}((G, \sigma)) - 1 \leq \chi((G, \sigma)) \leq \chi_{\pm}((G, \sigma)) + 1$.*

Proof. Let $\chi_{\pm}((G, \sigma)) = n$ and c be an n-coloring of (G, σ) with colors from M_n.

If $n = 2k + 1$, then let $\phi : M_{2k+1} \to \mathbb{Z}_{2k+1}$ with $\phi(t) = t$ if $t \in \{0, \dots, k\}$, and $\phi(t) = 2k + 1 + t$ if $t \in \{-k, \dots, -1\}$. Then c is a signed $(2k + 1)$-coloring of (G, σ) with colors from M_{2k+1} if and only if $\phi \circ c$ is a $(2k + 1)$-coloring of (G, σ). Hence, $\chi((G, \sigma)) \leq \chi_{\pm}((G, \sigma))$. If $n = 2k$, then let $\phi' : M_{2k} \to \mathbb{Z}_{2k+1}$ with $\phi(t) = t$ if $t \in \{1, \dots, k\}$ and $\phi(t) = 2k + 1 + t$ if $t \in \{-k, \dots, -1\}$. Then $\phi' \circ c$ is a $(2k + 1)$-coloring of (G, σ). Hence, $\chi((G, \sigma)) \leq \chi_{\pm}((G, \sigma)) + 1$.

We analogously deduce that $\chi_{\pm}((G, \sigma)) \leq \chi((G, \sigma)) + 1$. □

Let G be a graph and $\Sigma(G)$ be the set of pairwise non-equivalent signatures on G.

The chromatic spectrum of G is the set $\{\chi((G, \sigma)) : \sigma \in \Sigma(G)\}$, which is denoted by $\Sigma_{\chi}(G)$. Analogously, the signed chromatic spectrum of G is the set $\{\chi_{\pm}((G, \sigma)) : \sigma \in \Sigma(G)\}$. It is denoted by $\Sigma_{\chi_{\pm}}(G)$. Let $M_{\chi}(G) = \max\{\chi((G, \sigma)) : \sigma \in \Sigma(G)\}$ and $m_{\chi}(G) = \min\{\chi((G, \sigma)) : \sigma \in \Sigma(G)\}$. Analogously, $M_{\chi_{\pm}}(G) = \max\{\chi_{\pm}((G, \sigma)) : \sigma \in \Sigma(G)\}$ and $m_{\chi_{\pm}}(G) = \min\{\chi_{\pm}((G, \sigma)) : \sigma \in \Sigma(G)\}$.

The following theorems are our main results.

Theorem 1.2. *If G is a graph, then $\Sigma_{\chi}(G) = \{k : m_{\chi}(G) \leq k \leq M_{\chi}(G)\}$.*

Theorem 1.3. *If G is a graph, then $\Sigma_{\chi_{\pm}}(G) = \{k : m_{\chi_{\pm}}(G) \leq k \leq M_{\chi_{\pm}}(G)\}$.*

Theorems 1.2 and 1.3 will be proved in Sections 2 and 3, respectively.

2. The chromatic spectrum of a graph

We start with the determination of $m_{\chi}(G)$.

Proposition 2.1. *Let G be a nonempty graph. The following statements hold.*

1. *$\Sigma_{\chi}(G) = \{1\}$ if and only if $m_{\chi}(G) = 1$ if and only if $E(G) = \emptyset$.*
2. *If $E(G) \neq \emptyset$, then $\Sigma_{\chi}(G) = \{2\}$ if and only if $m_{\chi}(G) = 2$ if and only if G is bipartite.*
3. *If G is not bipartite, then $m_{\chi}(G) = 3$.*

Proof. Statements 1 and 2 are obvious. For statement 3 consider (G, σ) where σ is the signature with all edges negative. Then $c : V(G) \to \mathbb{Z}_3$ with $c(v) = 1$ is a 3-coloring of G. Since G is not bipartite the statement follows with statements 1 and 2. □

If (G, σ) is a signed graph and $u \in V(G)$, then σ_u denotes the restriction of σ to $G - u$. A k-chromatic signed graph (G, σ) is k-chromatic critical if $\chi((G - u, \sigma_u)) < k$, for every $u \in V(G)$. The following proposition states some basic facts on k-chromatic critical graphs. The complete graph on n vertices is denoted by K_n.

Proposition 2.2. *Let (G, σ) be a signed graph.*

1. *(G, σ) is 1-critical if and only if $G = K_1$.*
2. *(G, σ) is 2-critical if and only if $G = K_2$.*
3. *(G, σ) is 3-critical if and only if G is an odd circuit.*

Proof. Statements 1 and 2 are obvious. An odd circuit with any signature is 3-critical. For the other direction let G be a 3-critical graph. Note, that (*) $G - u$ is bipartite for every $u \in V(G)$ by Proposition 2.1. Since G is not bipartite it follows that every vertex of G is contained in all odd circuits of G, and by (*) every odd circuit C is hamiltonian. C cannot contain a chord, since for otherwise G contains a non-hamiltonian odd circuit, a contradiction. Hence, G is an odd circuit. □

Lemma 2.3. *Let $k \geq 1$ be an integer. If (G, σ) is k-chromatic, then $\chi((G - u, \sigma_u)) \in \{k, k - 1\}$, for every $u \in V(G)$. In particular, if (G, σ) is k-critical, then $\chi((G - u, \sigma_u)) = k - 1$.*