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Book embedding of locally planar graphs on orientable surfaces

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ABSTRACT

A *book embedding* of a graph G is an embedding of vertices of G along the *spine* of a book, and edges of G on the *pages* so that no two edges on the same page intersect. Malitz (1994) proved that any graph on the orientable surface $\mathbb{S}_g$ of genus g has a book embedding with $O(\sqrt{g})$ pages. In this paper, we prove that every *locally planar* graph on $\mathbb{S}_g$ (i.e., one with high representativity) has a book embedding with seven pages.

1. Introduction

A *book* (or a *book with k pages*) is a 3-dimensional structure consisting of a line, called a *spine*, and k distinct half-planes, called *pages*, having the spine as their boundaries. A *book embedding* of a graph G is an embedding of G into a book such that the vertices of G are represented as distinct points of the spine and that each edge of G is a circular arc lying in a single page, with the requirement that any two edges on the same page do not intersect. For example, see Fig. 1. A graph G is *k-page embeddable* if G admits an embedding into a book with at most k pages. The *pagenumber* of a graph G, denoted $p(G)$, is the minimum number of k such that G is k-page embeddable. (The page number is also called a *book thickness* or a *stack number*.) Book embeddings find application in fault tolerant multiprocessing including VLSI design [4].

For several classes of graphs, the pagenumber is known. First, the pagenumber of a complete graph K_n with n vertices is given in [4] by:

$$p(K_n) = \left\lceil \frac{n}{2} \right\rceil \quad (n \geq 4). \tag{1}$$

In addition, upper bounds for the pagenumber of several graph classes are known, for example, complete bipartite graphs [7,17] and k-trees [5,8,21]. In particular, 1-page embeddable graphs and 2-page embeddable graphs are completely characterized as follows:

Proposition 1 (*Bernhart and Kainen [2]*).

(i) *A graph G is* 1*-page embeddable if and only if G is outerplanar, and*
(ii) *a graph G is* 2*-page embeddable if and only if G is a subgraph of a Hamiltonian planar graph.*

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By the above proposition, every graph embeddable in a book with one or two pages is a special kind of planar graph. In this paper, we consider the pagenumber of graphs on surfaces.

By a *surface*, we mean a connected compact 2-dimensional manifold without boundary. By the classification of surfaces, every surface is homeomorphic to either

- a sphere with $2g$ disjoint disks $D_1, D'_1, \dots, D_g, D'_g$ removed, and a single handle attached to join D_i and D'_i for $i = 1, \dots, g$, for some $g \geq 0$, or
- a sphere with k disjoint disks $D_1, \dots, D_k$ removed and replaced with k Möbius bands, respectively, for some $k > 0$,

where the former is the *orientable* surface of genus g, denoted $\mathbb{S}_g$, and the latter the *nonorientable surface* of genus k, denoted $\mathbb{N}_k$. (The readers should refer to [16] for the detailed definitions.) In particular, $\mathbb{S}_0$ and $\mathbb{S}_1$ are the *sphere* and the *torus*, respectively.

Bernhart and Kainen conjectured that the pagenumber of planar graphs can be arbitrarily large [2]. However, Buss and Shor [3] disproved the conjecture, proving that every planar graph is 9-page embeddable. After that, Heath [9] improved the bound to 7. Finally, Yannakakis [22] proved that every planar graph is 4-page embeddable. However, it is still unknown whether there exists a planar graph requiring four pages.

For the torus $\mathbb{S}_1$, Endo [6] proved that every *toroidal* graph (i.e., one embeddable in the torus) has a 7-page book embedding. It is well-known that K_7 is embeddable in the torus, and $p(K_7) = 4$ from (1). However, we do not know any toroidal graph which requires more than four pages, and we do not know whether or not the upper bound "7" is best possible. In general, Heath and Istrail [10] proved that every graph G embeddable in $\mathbb{S}_g$ has pagenumber $O(g)$, and later Malitz [15] improved this bound into $O(\sqrt{g})$, where this bound is best possible, as follows: It was shown in [18] that $\mathbb{S}_g$ admits an embedding of K_n with $n = \lfloor \frac{7+\sqrt{48g+1}}{2} \rfloor$. On the other hand, $p(K_n) = \lceil \frac{n}{2} \rceil$ from (1). Hence there exists a graph $G = K_n$ embeddable in $\mathbb{S}_g$ with $p(G) = \lceil \frac{1}{2}(\lfloor \frac{7+\sqrt{48g+1}}{2} \rfloor) \rceil = \Theta(\sqrt{g})$.

Let $\mathbb{F}$ be a non-spherical surface. A simple closed curve γ on $\mathbb{F}$ is *contractible* if γ bounds a 2-cell. The *representativity* of a graph G on $\mathbb{F}$, denoted $r(G)$, is the minimum number of intersecting points of G and γ on $\mathbb{F}$, where γ ranges over all non-contractible simple closed curves on $\mathbb{F}$. A *locally planar* graph is a graph on a non-spherical surface with sufficiently large representativity. (Here "sufficiently large" will depend on the surface as well as the property being sought. So "locally planar graphs on $\mathbb{F}$ satisfy a property $\mathcal{P}$" means that there exists an integer $N(\mathbb{F})$ such that every graph on $\mathbb{F}$ with representativity at least $N(\mathbb{F})$ satisfies $\mathcal{P}$.) It is known that locally planar graphs often have a similar property to that of planar graphs, independent of the genus. For example, we know that every planar graph is 4-colorable [1], but some graphs on $\mathbb{S}_g$ might require $\lfloor \frac{7+\sqrt{48g+1}}{2} \rfloor$ colors [18]. However, Thomassen [20] proved that locally planar graphs on any non-spherical surface $\mathbb{F}$ are 5-colorable, independent of the genus of $\mathbb{F}$. See [12,13] for other results on graph coloring, [14] for spanning subgraphs, and [11] for graph domination.

In this paper, we investigate whether a locally planar graph on $\mathbb{S}_g$ has a book embedding with a constant number of pages. Recall that graphs on $\mathbb{S}_g$ require $\Theta(\sqrt{g})$ pages in general [15]. The following is our main theorem:

Theorem 2. *For the orientable surface $\mathbb{S}_g$, there exists a positive integer $\mathbb{N}(\mathbb{S}_g)$ such that every graph on $\mathbb{S}_g$ with representativity at least $\mathbb{N}(\mathbb{S}_g)$ has a 7-page embedding.*

In Section 2, we introduce the notation and terminology, and provide preliminary results which are used in the proof of Theorem 2. In Section 3, we describe techniques from [14] explaining how to cut a locally planar graph on $\mathbb{S}_g$ into a plane graph, and we prove Theorem 2 in Section 4.

2. Terminology and preliminary results

Let G be a graph, where we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. For $v \in V(G)$, the *neighborhood* of v, denoted by $N(v)$, is the set of all vertices adjacent to v. Cycles (or paths) $Q_1, \dots, Q_t$ of G are *disjoint* if $V(Q_i) \cap V(Q_j) = \emptyset$ for any distinct $i, j \in \{1, \dots, t\}$. Let Q be a sequence of vertices of G. For $u, v \in Q$, we denote the subsequence of Q from u to v by $Q[u, v]$. In addition, $Q(u, v]$ is the subsequence of Q obtained from $Q[u, v]$ by deleting the vertex u. Similarly, we define the subsequences $Q[u, v)$ and $Q(u, v)$ of Q. For two sequence of vertices $Q_1 = x_0x_1 \cdots x_k$ and $Q_2 = y_0y_1 \cdots y_l$, we denote the sequence obtained by the concatenation of Q_1 and Q_2 by Q_1Q_2, that is, $Q_1Q_2 = x_0x_1 \cdots x_ky_0y_1 \cdots y_l$. In this paper, we regard a path or an oriented cycle in G also as a sequence of vertices.

For a book embedding Σ of a graph G, the sequence of the vertices of G on the spine is the *spine sequence* σ of Σ. For a subgraph H of G, *the spine subsequence* $\sigma_{V(H)}$ is the subsequence of σ obtained from σ by deleting all vertices in $V(G) - V(H)$. Let G be a connected *plane graph*, that is, a graph drawn in the plane with no edge crossing. The *outer walk* of G is the closed walk bounding the infinite face of G. In particular, if the outer walk is simple (i.e., with no repeated vertices), then it is the *outer cycle* of G, denoted by ∂G.

A *near triangulation* is a 2-connected plane graph where all finite faces are triangular. We decompose a triangulation on a non-spherical surface into several near triangulations, using the following proposition. (We can easily see that Proposition 3 holds, and so we omit its proof.)