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The asymptotic behavior of the correspondence chromatic number

Anton Bernshteyn

University of Illinois at Urbana-Champaign, United States

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ABSTRACT

Alon (2000) proved that for any graph G, $\chi_\ell(G) = \Omega(\ln d)$, where $\chi_\ell(G)$ is the list chromatic number of G and d is the average degree of G. Dvořák and Postle (2015) recently introduced a generalization of list coloring, which they called *correspondence coloring*. We establish an analog of Alon's result for correspondence coloring; namely, we show that $\chi_c(G) = \Omega(d/\ln d)$, where $\chi_c(G)$ denotes the correspondence chromatic number of G. We also prove that for triangle-free G, $\chi_c(G) = O(\Delta/\ln \Delta)$, where Δ is the maximum degree of G (this is a generalization of Johansson's result about list colorings (Johansson, 1996)). This implies that the correspondence chromatic number of a regular triangle-free graph is, up to a constant factor, determined by its degree.

1. Introduction

An important generalization of graph coloring, so-called *list coloring*, was introduced independently by Vizing [11] and Erdős, Rubin, and Taylor [6]. It is defined as follows. Let G be a graph[1] and suppose that for each vertex $v \in V(G)$, a set of available colors $L(v)$, called the *list* of v, is specified. A proper coloring c of G is an L-*coloring* if $c(v) \in L(v)$ for all $v \in V(G)$. G is said to be L-*colorable* if it admits an L-coloring; G is k-*list-colorable* (or k-*choosable*) if it is L-colorable whenever $|L(v)| \geq k$ for all $v \in V(G)$. The least number k such that G is k-choosable is called the *list chromatic number* (or the *choosability*) of G and is denoted by $\chi_\ell(G)$ (or $\mathrm{ch}(G)$).

For all graphs G, $\chi(G) \leq \chi_\ell(G)$, where $\chi(G)$ denotes the ordinary chromatic number of G. Indeed, G is k-colorable if and only if it is L-colorable with the list assignment such that $L(v) = \{1, \ldots, k\}$ for all $v \in V(G)$. This inequality can be strict; in fact, $\chi_\ell(G)$ cannot be bounded above by any function of $\chi(G)$ since there exist bipartite graphs with arbitrarily high list chromatic numbers.

A striking difference between list coloring and ordinary coloring was observed by Alon in [1]: It turns out that the list chromatic number of a graph can be bounded below by an increasing function of its average degree. More precisely:

Theorem 1.1 (*Alon [1]*). *Let G be a graph with average degree d. Then*

$$\chi_\ell(G) \geq (1/2 - o(1)) \log_2 d,$$

where we assume that $d \to \infty$.

E-mail address: bernsht2@illinois.edu.

[1] All graphs considered here are finite, undirected, and simple.

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In this paper we study another notion, more general than list coloring, which was recently introduced by Dvořák and Postle [5] in order to prove that every planar graph without cycles of lengths 4–8 is 3-list-colorable, answering a long-standing question of Borodin [4]. Before we proceed to the actual definition, let us consider an example. Suppose that G is a graph and L is a list assignment for G. For each $v \in V(G)$, let

$$\widetilde{L}(v) := \{(v, c) \, : \, c \in L(v)\}.$$

Thus, the sets $\widetilde{L}(v)$ are pairwise disjoint. Let H be the graph with vertex set

$$V(H) := \bigcup_{v \in V(G)} \widetilde{L}(v)$$

and edge set

$$E(H) := \{(v_1, c_1)(v_2, c_2) \, : \, v_1 v_2 \in E(G), c_1 = c_2\}.$$

Given an L-coloring c of G, we define the set $I_c \subseteq V(H)$ as follows:

$$I_c := \{(v, c(v)) \, : \, v \in V(G)\}.$$

Observe that I_c is an independent set in H and for each vertex $v \in V(G)$, $|I_c \cap \widetilde{L}(v)| = 1$. Conversely, if $I \subseteq V(H)$ is an independent set such that $|I \cap \widetilde{L}(v)| = 1$ for all $v \in V(G)$, then, setting $c_I(v)$ to be the single color such that $(v, c_I(v)) \in I$, we obtain a proper L-coloring c_I of G.

This example can be generalized as follows.

Definition 1.2. Let G be a graph. A *cover* of G is a pair (L, H), where L is an assignment of pairwise disjoint sets to the vertices of G and H is a graph with vertex set $\bigcup_{v \in V(G)} L(v)$, satisfying the following two conditions.

1. If $xy \in E(H)$, where $x \in L(u)$, $y \in L(v)$, then $uv \in E(G)$ (in particular, $u \neq v$).
2. For each $uv \in E(G)$, the edges between $L(u)$ and $L(v)$ form a matching.

Definition 1.3. Suppose that G is a graph and (L, H) is a cover of G. An *(L, H)-coloring* of G is an independent set $I \subseteq V(H)$ such that $I \cap L(v) \neq \emptyset$ for all $v \in V(G)$. In this context, we refer to the vertices of H as the *colors*. G is said to be *(L, H)-colorable* if it admits an (L, H)-coloring.

Remark 1.4. Note that Definition 1.3 allows for more than one color to be used at a given vertex of G. However, if G is (L, H)-colorable, then we can always find an (L, H)-coloring that uses exactly one color for each vertex.

Definition 1.5. A graph G is *k-correspondence-colorable* (*k-c.c.* for short) if it is (L, H)-colorable whenever (L, H) is a cover of G and $|L(v)| \geq k$ for all $v \in V(G)$. The least number k such that G is k-c.c. is called the *correspondence chromatic number* of G and is denoted by $\chi_c(G)$.

The above example shows that $\chi_\ell(G) \leq \chi_c(G)$ for all graphs G. As in the case of ordinary vs. list chromatic number, this inequality can be strict. For instance, if C_{2n} is the cycle of length $2n$, then $\chi_\ell(C_{2n}) = 2$, while $\chi_c(C_{2n}) = 3$. Nevertheless, several known upper bounds for list coloring can be transferred to the correspondence coloring setting. For example, it is not hard to show that $\chi_c(G) \leq \Delta + 1$ for any graph G with maximum degree Δ. Dvořák and Postle observed in [5] that $\chi_c(G) \leq 5$ if G is planar, and $\chi_c(G) \leq 3$ if G is planar and has girth at least 5; these bounds are the analogs of Thomassen's results about list colorings [9,10].

Our first result is an analog of Theorem 1.1 for correspondence chromatic number.

Theorem 1.6. *Let G be a graph with average degree $d \geq 2e$. Then*

$$\chi_c(G) \geq \frac{d/2}{\ln(d/2)}.$$

Theorem 1.6 shows that the correspondence chromatic number of a graph grows with the average degree much faster than the list chromatic number and, in fact, is only a logarithmic factor away from the trivial upper bound $\chi_c(G) \leq d + 1$ for d-regular G.

A celebrated theorem of Johansson [7] asserts that for any triangle-free graph G with maximum degree Δ, $\chi_\ell(G) = O(\Delta / \ln \Delta)$. Our next result shows that the same upper bound holds for correspondence coloring as well.

Theorem 1.7. *There exists a positive constant C such that for any triangle-free graph G with maximum degree Δ,*

$$\chi_c(G) \leq C \frac{\Delta}{\ln \Delta}.$$

Combining Theorems 1.6 and 1.7, we immediately obtain the following corollary.