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A zero-free interval for chromatic polynomials of graphs with 3-leaf spanning trees

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ABSTRACT

It is proved that if G is a graph containing a spanning tree with at most three leaves, then the chromatic polynomial of G has no roots in the interval $(1, t_1]$, where $t_1 \approx 1.2904$ is the smallest real root of the polynomial $(t-2)^6+4(t-1)^2(t-2)^3-(t-1)^4$. We also construct a family of graphs containing such spanning trees with chromatic roots converging to t_1 from above. We employ the Whitney 2-switch operation to manage the analysis of an infinite class of chromatic polynomials.

1. Introduction

The *chromatic polynomial* $P(G, t)$ of a graph G is a polynomial with integer coefficients which counts, for each non-negative integer t, the number of t-colourings of G. It was introduced by Birkhoff [2] in 1912 for planar graphs, and extended to all graphs by Whitney [9,10] in 1932. If t is a real number then we say t is a *chromatic root* of G if $P(G, t) = 0$. Thus the numbers $0, 1, 2, \ldots, \chi(G) - 1$ are always chromatic roots of G and, in fact, the only rational ones. On the other hand, it is easy to see that the interval $(-\infty, 0)$ contains no chromatic roots, and Tutte [8] showed that the same is true for the interval $(0, 1)$. We say that such intervals are *zero-free* for the class of all graphs. In 1993, Jackson [5] proved the surprising result that the interval $(1, 32/27]$ is also zero-free, and found a sequence of graphs whose chromatic roots converge to $32/27$ from above. Thomassen [6] strengthened this by showing that the set of chromatic roots consists of 0, 1, and a dense subset of the interval $(32/27, \infty)$.

Let $Q(G, t) = (-1)^{|V(G)|}P(G, t)$, and $b(G)$ be the number of blocks of G. We say that G is *separable* if $b(G) \geq 2$ and *non-separable* otherwise. Note that K_2 is non-separable. In [7], Thomassen provided a new link between Hamiltonian paths and colourings by proving that the zero-free interval of Jackson can be extended when G has a Hamiltonian path. More precisely he proved the following.

Theorem 1.1 ([7]). *If G is a non-separable graph with a Hamiltonian path, then $Q(G, t) > 0$ for $t \in (1, t_0]$, where $t_0 \approx 1.295$ is the unique real root of the polynomial $(t-2)^3 + 4(t-1)^2$. Furthermore, for all $\varepsilon > 0$ there exists a non-separable graph with a Hamiltonian path whose chromatic polynomial has a root in the interval $(t_0, t_0 + \varepsilon)$.*

If G is separable and has a Hamiltonian path, then it is easily seen using Theorem 1.1 and Proposition 2.2 that $Q(G, t)$ is non-zero in the interval $(1, t_0]$ with sign $(-1)^{b(G)-1}$.

For a graph G, a *k-leaf spanning tree* is a spanning tree of G with at most k leaves (vertices of degree 1). We denote the class of non-separable graphs which admit a k-leaf spanning tree by $\mathcal{G}_k$. Thus, Theorem 1.1 gives a zero-free interval for the class $\mathcal{G}_2$. In this article we prove the following analogous result for the class $\mathcal{G}_3$.

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Theorem 1.2. *If G is a non-separable graph with a 3-leaf spanning tree, then $Q(G,t) > 0$ for $t \in (1, t_1]$, where $t_1 \approx 1.2904$ is the smallest real root of the polynomial $(t-2)^6 + 4(t-1)^2(t-2)^3 - (t-1)^4$. Furthermore, for all $\varepsilon > 0$, there exists a non-separable graph with a 3-leaf spanning tree whose chromatic polynomial has a root in the interval $(t_1, t_1 + \varepsilon)$.*

A natural extension of this work would be to find $\varepsilon_k > 0$ so that $(1, 32/27 + \varepsilon_k]$ is zero-free for the class $\mathcal{G}_k$, $k \geq 4$. However since the graphs presented by Jackson in [5] are non-separable, it must be that $\varepsilon_k \to 0$ as $k \to \infty$. Another possible extension would be to find $\varepsilon_\ell > 0$ so that $(1, 32/27 + \varepsilon_\ell]$ is zero-free for the family of graphs containing a spanning tree T with $\Delta(T) \leq 3$ and at most ℓ vertices of degree 3. Here the possible implications are much more interesting since it is not clear if $\varepsilon_\ell \to 0$ as $\ell \to \infty$. Indeed a short argument shows that only finitely many of the graphs presented by Jackson in [5] have a spanning tree of maximum degree 3. Theorem 1.1 and our result solve the cases $\ell = 0$ and $\ell = 1$ respectively, which leads us to conjecture the following.

Conjecture 1.1. *There exists $\varepsilon > 0$ such that if G is a non-separable graph with a spanning tree of maximum degree 3, then $Q(G,t) > 0$ for $t \in (1, 32/27 + \varepsilon]$.*

Barnette [1] proved that a 3-connected planar graph has a spanning tree of maximum degree 3. Thus an affirmative answer to Conjecture 1.1 would immediately imply a zero-free interval for the class of 3-connected planar graphs. Such an interval was found by Dong and Jackson [3] but it is thought to be far from maximal.

2. Preliminaries

All graphs in this article are *simple*, that is they have no loops or multiple edges. If u and v are vertices of G, then G/uv denotes the graph obtained by deleting the edge uv if it exists, and then identifying the vertices u and v. This operation is referred to as the *contraction* of uv. If G is connected, $S \subset V(G)$, and $G - S$ is disconnected, then S is called a *cut-set* of G. A 2-*cut* of G is a cut-set S with $|S| = 2$. If S is a cut-set of G and C is a component of $G - S$, then we say the graph $G[V(C) \cup S]$ is an S-*bridge* of G. Finally, if P is a path and $x, y \in V(P)$ then $P[x,y]$ denotes the subpath of P from x to y.

We make repeated use of two fundamental results in the study of chromatic polynomials.

Proposition 2.1 (*Deletion–contraction identity*)**.** *If G is a graph and uv is an edge of G, then*

$$P(G,t) = P(G - uv, t) - P(G/uv, t).$$

Proposition 2.2 (*Factoring over complete subgraphs*)**.** *If $G = G_1 \cup G_2$ be a graph such that $G[V(G_1) \cap V(G_2)]$ is a complete graph on r vertices, then*

$$P(G,t) = \frac{P(G_1,t)P(G_2,t)}{P(K_r,t)}.$$

The next proposition is easily proven using Propositions 2.1 and 2.2. The operation involved is often called a *Whitney 2-switch.*

Proposition 2.3. *Let G be a graph and $\{x,y\}$ be a 2-cut of G. Let C denote a connected component of $G - \{x,y\}$. Define G' to be the graph obtained from the disjoint union of $G - C$ and C by adding, for all $z \in V(C)$, the edge xz (respectively yz) if and only if yz (respectively xz) is an edge of G. Then we have $P(G,t) = P(G',t)$.*

If G' can be obtained from G by a sequence of Whitney 2-switches, then $P(G,t) = P(G',t)$ and we say G and G' are *Whitney equivalent.*

Definition 2.1. A graph G is a generalised triangle if the following conditions hold:

- G is non-separable but not 3-connected.
- For every 2-cut $\{x,y\}$, $xy \notin E(G)$ and G has precisely three $\{x,y\}$-bridges, all of which are separable.

The class of generalised triangles was first defined by Jackson in [5] and is an important class of graphs in the study of chromatic roots. The name is derived from an equivalent characterisation, which says that the generalised triangles are the graphs that can be obtained from K_3 by repeatedly replacing an edge uv by two paths of length 2 with ends u and v, see Jackson [5].

2.1. Hamiltonian paths

We briefly describe a number of results, quantities, and definitions from Thomassen [7], as they will play an important role in our result.