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Dominating sets inducing large components

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ABSTRACT

As a common generalization of the domination number and the total domination number of a graph G, we study the k-component domination number $\gamma_k(G)$ of G defined as the minimum cardinality of a dominating set D of G for which each component of the subgraph $G[D]$ of G induced by D has order at least k.

We show that for every positive integer k, and every graph G of order n at least $k+1$ and without isolated vertices, we have $\gamma_k(G) \leq \frac{kn}{k+1}$. Furthermore, we characterize all extremal graphs. We propose two conjectures concerning graphs of minimum degree 2, and prove a related result.

1. Introduction

The two most prominent domination parameters [7,10] are the domination number $\gamma(G)$ and the total domination number $\gamma_t(G)$ of a graph G. Several analogous results were established independently for these two parameters. In the present paper we consider the k-component domination number $\gamma_k(G)$ of a graph G as a common generalization. Our goals are unified results and proofs that generalize statements obtained separately for $\gamma(G)$ and $\gamma_t(G)$.

A set D of vertices of a graph G is *dominating* if every vertex of G that is not in D has a neighbor in D. For some positive integer k, the set D is a *k-component dominating set* of G if it is dominating and every component of the subgraph $G[D]$ of G induced by D has order at least k. The *k-component domination number* $\gamma_k(G)$ of G is the minimum order of a k-component dominating set of G. Note that a graph has a k-component dominating set if and only if each of its components has order at least k. Clearly, $\gamma_1(G) = \gamma(G)$ and $\gamma_2(G) = \gamma_t(G)$. For a non-negative integer k, let $[k]$ be the set of the positive integers at most k. For a graph G with vertex set $\{u_1, \ldots, u_n\}$ and a positive integer k, let the graph $G \circ P_k$ arise from the disjoint union of G and n copies $P^{(1)}, \ldots, P^{(n)}$ of the path P_k of order k, by adding an edge between u_i and one endvertex of $P^{(i)}$ for every $i \in [n]$. The cycle of order n is denoted C_n.

There is already one nice example for the kind of result we are seeking. For a maximal outerplanar graph G of order n at least 5, Matheson and Tarjan [12] showed $\gamma(G) \leq \lfloor \frac{n}{3} \rfloor$, and Dorfling, Hattingh, and Jonck [5] recently showed $\gamma_t(G) \leq \lfloor \frac{2n}{5} \rfloor$ unless G is one of two specific exceptional graphs of order 12. In view of the completely different approaches used to prove these two results and also in view of the exceptional graphs that appear only for total domination, it was surprising that we were [1] able to obtain a common generalization with a unified proof. If k and n are positive integers with $n \geq 2k+1$, and G is a maximal outerplanar graph of order n, we showed [1]

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$$\gamma_k(G) \le \begin{cases} \left\lceil \dfrac{kn}{2k+1} \right\rceil, & \text{if } G \in \mathcal{H}_k \\ \left\lfloor \dfrac{kn}{2k+1} \right\rfloor, & \text{otherwise,} \end{cases}$$

where $\mathcal{H}_k$ is a set of well-described graphs each of order at least $4k+4$ and at most $4k^2-2k$. Note that the bounds on the orders necessarily imply that $\mathcal{H}_1$ is empty.

In the present paper we consider some of the most classical results in domination theory, upper bounds that depend on the order and the minimum degree of the considered graphs as well as the extremal graphs.

2. Results

The following result summarizes several contributions relevant for graphs of minimum degree at least 1. For convenience they are phrased in terms of connected graphs.

Theorem 1. *Let G be a connected graph of order n.*

(i) *(Ore [14], Payan et al. [15], Fink et al. [6]) If $n \ge 2$, then $\gamma(G) \le \frac{1}{2}n$ with equality if and only if G either has order 2, or is C_4, or is $F \circ P_1$ for some connected graph F of order at least 2.*
(ii) *(Cockayne et al. [4], Brigham et al. [3]) If $n \ge 3$, then $\gamma_t(G) \le \frac{2}{3}n$ with equality if and only if G either has order 3, or is C_6, or is $F \circ P_2$ for some connected graph F of order at least 2.*

With Theorem 4 we give a common generalization of the two parts of Theorem 1. Similarly as for these two parts, an elegant proof of Theorem 4 relies on the existence of optimal dominating sets with additional properties. For a graph G, a set D of vertices of G, and a vertex u in D, let $\text{epn}_G(u,D) = N_G(u) \setminus \bigcup_{v \in D\setminus\{u\}} N_G[v]$ be the set of *external private neighbors of u in G with respect to D.*

Here are the relevant results for the domination number and the total domination number.

Theorem 2. *Let G be a connected graph of order n.*

(i) *(Bollobás et al. [2]) If $n \ge 2$, then G has a minimum dominating set D such that every vertex $u \in D$ satisfies $\text{epn}_G(u,D) \ne \emptyset$.*
(ii) *(Henning [8]) If $n \ge 3$ and G is not complete, then G has a minimum total dominating set D such that every vertex $u \in D$ satisfies $\text{epn}_G(u,D) \ne \emptyset$ or is adjacent to a vertex $v \in D$ with $d_{G[D]}(v) = 1$ and $\text{epn}_G(v,D) \ne \emptyset$.*

In order to extend Theorem 2 for $k \ge 3$, we need the following folklore lemma.

Lemma 1. *If G is a connected graph of order n such that every spanning tree of G is a path, then $G \in \{P_n, C_n\}$. In particular, every connected graph of order at least 3 that is not P_n, has at least 3 vertices that are not cutvertices.*

Proof. Let the path $P : u_1 \dots u_n$ be a spanning tree of G. If u_iu_j is an edge of G with $1 \le i < j < n$, then $P - u_iu_{i+1} + u_iu_j$ is a spanning tree of G with three endvertices u_1, u_n, and u_{i+1}. Hence, by symmetry, the only possible chord is u_1u_n, which implies $G \in \{P_n, C_n\}$. Since no vertex of a connected graph that is an endvertex of some spanning tree, is a cutvertex, the second statement follows from the first. □

Note that, if some vertex u in a minimum k-component dominating set D of a graph G does not have an external private neighbor with respect to D, then $D \setminus \{u\}$ is still a dominating set but some component of $G[D \setminus \{u\}]$ became too small.

We extend Theorem 2 for $k \ge 3$ as follows.

Theorem 3. *Let k be a positive integer at least 3. If G is a connected graph with $n(G) \ge k+2$ and $\gamma_k(G) \ge k+1$, then G has a minimum k-component dominating set D such that for $D_P = \{u \in D : \text{epn}_G(u,D) \ne \emptyset\}$, every component K of $G[D]$ satisfies $|V(K) \cap D_P| \ge \frac{n(K)}{k}$ with equality only if $K = F \circ P_{k-1}$ for some connected graph F, and $V(K) \cap D_P$ contains only endvertices of K.*

Proof. Let D be a minimum k-component dominating set of G such that

$$G[D] \text{ has as few components as possible.} \tag{1}$$

Suppose that K is a component of $G[D]$ of order k. Since $|D| \ge k+1$, the set $D \setminus V(K)$ is not empty. Let $P : u_0 \dots u_\ell$ be a shortest path between a vertex u_0 in $V(K)$ and a vertex u_ℓ in $D \setminus V(K)$. Note that the internal vertices of P do not lie in D. Suppose that $\ell \ge 4$. Since D is a dominating set, the vertex u_2 has a neighbor v in some component K' of $G[D]$. If $K = K'$, then $vu_2 \dots u_\ell$ is a shorter path than P between $V(K)$ and $D \setminus V(K)$, and, if $K \ne K'$, then $u_0u_1u_2v$ is a shorter path than P between $V(K)$ and $D \setminus V(K)$, which contradicts the choice of P. Hence, $\ell = \text{dist}_G(V(K), D \setminus V(K)) \in \{2,3\}$. Let $\kappa(D)$ be the number of components K of $G[D]$ of order k with $\text{dist}_G(V(K), D \setminus V(K)) = 3$. Subject to the condition (1), we choose D such that

$$\kappa(D) \text{ is as small as possible.} \tag{2}$$

Now, let K be a component of $G[D]$.