Discrete Mathematics 339 (2016) 2723–2726

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

A note on the values of independence polynomials at -1

Jonathan Cutler[a], Nathan Kahl[b,*]

[a] Department of Mathematical Sciences, Montclair State University, Montclair, NJ, United States

[b] Department of Mathematics and Computer Science, Seton Hall University, South Orange, NJ, United States

ARTICLE INFO

Article history:
Received 28 October 2014
Received in revised form 17 May 2016
Accepted 18 May 2016
Available online 11 June 2016

Keywords:
Independence polynomial
Decycling number

ABSTRACT

The *independence polynomial* $I(G; x)$ of a graph G is $I(G; x) = \sum_{k=0}^{\alpha(G)} s_k x^k$, where s_k is the number of independent sets in G of size k. The *decycling number* of a graph G, denoted $\phi(G)$, is the minimum size of a set $S \subseteq V(G)$ such that $G - S$ is acyclic. Engström proved that the independence polynomial satisfies $|I(G; -1)| \leq 2^{\phi(G)}$ for any graph G, and this bound is best possible. Levit and Mandrescu provided an elementary proof of the bound, and in addition conjectured that for every positive integer k and integer q with $|q| \leq 2^k$, there is a connected graph G with $\phi(G) = k$ and $I(G; -1) = q$. In this note, we prove this conjecture.

1. Introduction

Let $\alpha(G)$ denote the *independence number of a graph* G, the maximum order of an independent set of vertices in G. The *independence polynomial of a graph* G is given by

$$I(G; x) = \sum_{k=0}^{\alpha(G)} s_k x^k,$$

where s_k is the number of independent sets of size k in G. The independence polynomial has been the object of much research (see for instance the survey [7]). One direction of this research, partly motivated by connections with hard-particle models in physics [1–3,5,6], has focused on the evaluation of the independence polynomial at $x = -1$.

The *decycling number of a graph* G, denoted $\phi(G)$, is the minimum size of a set of vertices $S \subseteq V(G)$ such that $G - S$ is acyclic. Engström [3] proved the following bound on $I(G; -1)$, which is best possible.

Theorem 1.1 (*Engström*). *For any graph G, $|I(G; -1)| \leq 2^{\phi(G)}$.*

Levit and Mandrescu [8] gave an elementary proof of Theorem 1.1 and, in addition, proposed the following conjecture.

Conjecture 1 (*Levit and Mandrescu*). *Given a positive integer k and an integer q with $|q| \leq 2^k$, there is a connected graph G with $\phi(G) = k$ and $I(G; -1) = q$.*

For brevity, in this paper a graph G with $\phi(G) = k$ and $I(G; -1) = q$, with $|q| \leq 2^k$, will be referred to as a *(k, q)-graph*. In [9], Levit and Mandrescu provided constructions that gave (k, q)-graphs for all $k \leq 3$ and $|q| \leq 2^k$. Also, they gave constructions for every k provided $q \in \left\{2^{\phi(G)}, 2^{\phi(G)} - 1\right\}$. In this paper, we prove Conjecture 1.

* Corresponding author.
E-mail addresses: jonathan.cutler@montclair.edu (J. Cutler), nathan.kahl@shu.edu (N. Kahl).

http://dx.doi.org/10.1016/j.disc.2016.05.019
0012-365X/

Table 1
Some small examples.

G	$I(G; -1)$
K_1	0
K_2	-1
$K_3 = C_3$	-2
C_6	2

2. The construction and proof of conjecture

The construction proceeds inductively, using particular $(k-1, q)$-graphs to produce the necessary (k, q)-graphs. First we assemble the tools used in the construction. The most important tool is a recursive formula for $I(G; x)$ due to Gutman and Harary [4]. We let $N(v) = \{x \in V(G) : xv \in E(G)\}$ and $N[v] = \{v\} \cup N(v)$.

Lemma 2.1. *For any graph G and any vertex $v \in V(G)$,*

$$I(G; x) = I(G - v; x) + xI(G - N[v]; x).$$

Using this, or simply counting independent sets, we can derive the independence polynomial at -1 for small graphs. Some useful examples can be found in Table 1.

Since Lemma 2.1 requires a particular vertex $v \in V(G)$ to be specified, it will often be helpful to root graphs for which we want to compute the independence polynomial at -1. Given a graph G and a vertex $v \in V(G)$, the *rooted graph* G_v is the graph G with the vertex v labeled. Of course, $I(G; -1) = I(G_v; -1)$ for any vertex $v \in V(G)$.

We now introduce two operations on rooted graphs which will be useful in our proof. The first of these is called *pasting*.

Definition 1. Given two rooted graphs G_v and H_w, the *pasting of* G_v *and* H_w, denoted $G_v \wedge H_w$, is the rooted graph formed by identifying the roots v and w.

We note two important facts. First, the pasting operation creates no new cycles, and thus $\phi(G_v \wedge H_w) \le \phi(G_v) + \phi(H_w)$. (In our construction the roots will be pendant vertices, and so $\phi(G_v \wedge H_w) = \phi(G_v) + \phi(H_w)$.) Second, if for two rooted graphs G_v and H_w the quantities $I(G_v; -1)$ and $I(H_w; -1)$ have been evaluated using Lemma 2.1, then the value of $I(G_v \wedge H_w; -1)$ can be determined in a straightforward way. It is well-known that, letting $G \cup H$ denote the disjoint union of G and H, we have

$$I(G \cup H; x) = I(G; x)I(H; x).$$

Deleting the pasted vertex in $G_v \wedge H_w$ produces a disjoint union of graphs. This fact, and the recurrences

$$I(G_v; -1) = I(G_v - v; -1) - I(G_v - N[v]; -1)$$
$$I(H_w; -1) = I(H_w - w; -1) - I(H_w - N[w]; -1)$$

then give

$$I(G_v \wedge H_w; -1) = I(G_v - v; -1)I(H_w - w; -1) - I(G_v - N[v]; -1)I(H_w - N[w]; -1).$$

It will be helpful to keep track of the various parts of the above calculation, and in order to do so we introduce the following bookkeeping device. Given a rooted graph G_v, where $I(G_v - v; -1) = a$ and $I(G_v - N[v]; -1) = b$, and hence $I(G_v; -1) = a - b$, we write $I(G_v; -1) = \langle a, b \rangle$ and say that G_v has *bracket* $\langle a, b \rangle$. An example can be found in Fig. 1. Note that for a given rooted graph G_v there are unique integers a and b, determined by the root, with $I(G_v; -1) = \langle a, b \rangle$. Using this notation, the calculations above give the following lemma.

Lemma 2.2 (*Pasting Lemma*). *If G_v and H_w are rooted graphs on at least two vertices with $I(G_v; -1) = a - b = \langle a, b \rangle$ and $I(H_w; -1) = c - d = \langle c, d \rangle$, then*

$$I(G_v \wedge H_w; -1) = ac - bd = \langle ac, bd \rangle.$$

Our second operation is a variation of the pasting operation which, however, is useful enough to merit its own terminology and notation.

Definition 2. Given a rooted graph G_v and an integer $k \ge 0$, the *ℓ-extension of* G_v, denoted G_v^ℓ is the graph formed by identifying the root v with one of the endpoints of a (disjoint) path of length ℓ and reassigning the root to the other endpoint of the path.