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Note

A note on tilted Sperner families with patterns

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ABSTRACT

Let p and q be two nonnegative integers with $p+q>0$ and $n>0$. We call $\mathcal{F}\subset\mathcal{P}([n])$ a *(p,q)-tilted Sperner family with patterns* on $[n]$ if there are no distinct $F,G\in\mathcal{F}$ with:

(i) $p|F\setminus G|=q|G\setminus F|$, and
(ii) $f>g$ for all $f\in F\setminus G$ and $g\in G\setminus F$.

E. Long in Long (2015) proved that the cardinality of a $(1,2)$-tilted Sperner family with patterns on $[n]$ is

$$O\left(e^{120\sqrt{\log n}}\frac{2^n}{\sqrt{n}}\right).$$

We improve and generalize this result, and prove that the cardinality of every (p,q)-tilted Sperner family with patterns on $[n]$ is

$$O\left(\sqrt{\log n}\,\frac{2^n}{\sqrt{n}}\right).$$

1. Introduction

A family $\mathcal{F}$ of subsets of $[n]$ (where for $n>0$ we will use the $[n]$ notation for $\{1,2,\ldots,n\}$ and $\mathcal{P}([n])$ for the power set) is called a *Sperner family* if $F\not\subset G$ for all distinct $F,G\in\mathcal{F}$. A classic result in extremal combinatorics is Sperner's theorem [12], which states that the maximal cardinality of a Sperner family is $\binom{n}{\lfloor\frac{n}{2}\rfloor}$. This result has a huge impact on combinatorics and has many generalizations (see e.g. [2]).

Recently Sperner's theorem played some role in the Polymath project to discover a new proof of the density Hales–Jewett theorem [11]. Motivated by its role in the proof Kalai asked whether one can achieve 'Sperner-like theorems' for 'Sperner like families' [8].

One direction to generalize the notion of Sperner families is the so called *tilted Sperner families* (see Definition 1.1). As written in [8]: Kalai noted that the 'no containment' condition can be rephrased as follows: $\mathcal{F}$ does not contain two sets F and G such that, in the unique subcube of $\mathcal{P}([n])$ spanned by F and G, the bottom point is F and G is the top point. He asked: what happens if we forbid F and G to be at a different position in this subcube? In particular, he asked how large $\mathcal{F}\subset\mathcal{P}([n])$ can be if we forbid F and G to be at a fixed ratio $p:q$ in this subcube. That is, we forbid F to be $p/(p+q)$ of the way up this subcube and G to be $q/(p+q)$ of the way up this subcube. Equivalently we can say:

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Definition 1.1. Let p, q be two nonnegative integers. We call $\mathcal{F} \subseteq \mathcal{P}([n])$ a *(p, q)-tilted Sperner family* if for all distinct $F, G \in \mathcal{F}$ we have

$$p|F \setminus G| \neq q|G \setminus F|.$$

Note that we can restrict ourselves to coprime p and q. Also note a Sperner family is just a $(1, 0)$-tilted Sperner family. In [8] Leader and Long proved the following theorem, which gives an asymptotically tight answer for the maximal cardinality of a (p, q)-tilted Sperner family:

Theorem 1.2. *Let p, q be coprime nonnegative integers with $q \geq p$. Suppose $\mathcal{F} \subset \mathcal{P}([n])$ is a (p, q)-tilted Sperner family. Then*

$$|\mathcal{F}| \leq (q - p + o(1))\binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Note that up to the $o(1)$ term, this is the best possible, since the union of $p - q$ consecutive levels is a (p, q)-tilted Sperner family.

In [10] Long started to investigate the cardinality of *tilted Sperner families with patterns* (see Definition 1.3), which was also asked by Kalai [9].

Definition 1.3. Let p and q be nonnegative integers with $p + q > 0$. We call $\mathcal{F}$ a *(p, q)-tilted Sperner family with patterns*, if there are no distinct $F, G \in \mathcal{F}$ with:

(i) $p|F \setminus G| = q|G \setminus F|$, and
(ii) $f > g$ for all $f \in F \setminus G$ and $g \in G \setminus F$.

In [10] he gave an upper bound on the cardinality of a $(1, 2)$-tilted Sperner family with patterns:

Theorem 1.4 (*[10, Theorem 1.3]*). *Let $\mathcal{F} \subset \mathcal{P}([n])$ be a $(1, 2)$-tilted Sperner family with patterns. Then*

$$|\mathcal{F}| \leq O\left(e^{120\sqrt{\log n}} \frac{2^n}{\sqrt{n}}\right).$$

Actually in [10] he gives a proof of a weaker result with the density Hales–Jewett theorem, and proves Theorem 1.4 with a randomized generalization of Katona's cycle method (see [6]).

In this note we generalize and improve his result by applying another generalization of Katona's cycle method, the so called permutation method. We will apply the permutation method in a somewhat similar way like the authors of [3] and prove the following:

Theorem 1.5. *Let p and q be nonnegative integers with $p + q > 0$ and let $\mathcal{F}$ be a (p, q)-tilted Sperner family with patterns. Then*

$$|\mathcal{F}| \leq O\left(\sqrt{\log n}\, \frac{2^n}{\sqrt{n}}\right).$$

The paper is organized as follows: in Section 2 we prove our main theorem and in Section 3 we pose some questions.

2. Proof of Theorem 1.5

Proof. If either p or q is zero, then we get back the usual Sperner family for which we know that the statement is true. In the following we fix $p, q > 0$ and furthermore we assume that $p \leq q$. The proof works similarly in case $p > q$.

2.1. The (p, q)-cut point

First we introduce a notion that will have crucial role in the proof.

Definition 2.1. We say that $x \in [n]$ is a *(p, q)-cut point of $A \subseteq [n]$*, if

$$0 \leq \frac{n - x - |([n] \setminus [x]) \cap A|}{q} - \frac{|A \cap [x]|}{p} < \frac{1}{p}. \tag{1}$$

We remark that x is a (p, q)-cut point means that $\frac{p}{q}$ times the number of points of A less than x is 'approximately' equal to the number of points not belonging to A that are larger than x.

Lemma 2.2. *Every $A \subseteq [n]$ has a (p, q)-cut point.*