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Plane graphs with maximum degree 9 are entirely 11-choosable[☆]

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ABSTRACT

A plane graph G is entirely k-choosable if, for every list L of colors satisfying $L(x) = k$ for all $x \in V(G) \cup E(G) \cup F(G)$, there exists a coloring which assigns to each vertex, each edge and each face a color from its list so that any adjacent or incident elements receive different colors. It was known that every plane graph G with maximum degree $\Delta \geq 10$ is entirely $(\Delta+2)$-choosable. In this paper, we improve this result by showing that every plane graph G with $\Delta = 9$ is entirely 11-choosable.

1. Introduction

All graphs considered in this paper are finite and simple. A *plane graph* is a particular drawing of a planar graph in the Euclidean plane. For a plane graph G, we denote its vertex set, edge set, face set, maximum degree, and minimum degree by $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$ (or V, E, F, Δ and δ if there is no confusion in the context), respectively.

A plane graph G is *entirely k-colorable* if $V \cup E \cup F$ can be colored with k colors such that any two adjacent or incident elements receive different colors. The *entire chromatic number* $\chi_{vef}(G)$ of G is defined to be the least integer k such that G is entirely k-colorable.

A mapping L is said to be an assignment for the plane graph G if it assigns a list $L(x)$ of possible colors to each element x in $V \cup E \cup F$. If G has an entire coloring ϕ such that $\phi(x) \in L(x)$ for all elements x, then we say that G is *entirely L-colorable* or ϕ is an *entire L-coloring* of G. G is *entirely k-choosable* if it is entirely L-colorable for every assignment L satisfying $|L(x)| = k$ for all elements x in $V \cup E \cup F$. The *entire choice number* $\mathrm{ch}_{vef}(G)$ of G is the smallest integer k such that G is entirely k-choosable.

In 1972, Kronk and Mitchem [8,9] proved that every plane graph G with $\Delta \leq 3$ is entirely $(\Delta+4)$-colorable, and conjectured that the result holds for any plane graph G with $\Delta \geq 4$. The upper bound $\Delta+4$ for $\chi_{vef}(G)$ is tight by observing that $\chi_{vef}(K_4) = 7 = \Delta(K_4) + 4$. Borodin confirmed the conjecture for $\Delta \geq 12$ in [1] and later improved to $\Delta \geq 7$ in [3]. In 2000, Sanders and Zhao [10] gave a proof for the case $\Delta = 6$, however their proof has a correctable error. In 2011, Wang and Zhu [17] settled the remaining cases $\Delta = 4, 5$, and corrected the Sanders–Zhao's proof on the case $\Delta = 6$.

Suppose that G is a simple plane graph. There has existed some deep discussion on the parameter $\chi_{vef}(G)$ and $\mathrm{ch}_{vef}(G)$ in recent years. First, recall that the celebrated Vizing's Theorem [11] says that the chromatic index $\chi'(G)$ of G is equal to Δ

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or $\Delta + 1$. Using this fact, Wang [13] showed that $\chi_{vef}(G) \leq \Delta + 4$ if $\chi'(G) = \Delta$, and $\chi_{vef}(G) \leq \Delta + 5$ if $\chi'(G) = \Delta + 1$. When Δ is larger, the upper bound on $\chi'_{vef}(G)$ may be replaced by $\Delta + 3$ or less. Wang, Mao and Miao [15] showed if $\Delta \geq 8$ then $\chi_{vef}(G) \leq \Delta + 3$. Borodin [2] proved that if $\Delta \geq 12$, then $\chi_{vef}(G) \leq \Delta + 2$ (in fact, his proof implies the stronger result that $\mathrm{ch}_{vef}(G) \leq \Delta + 2$). The upper bound $\Delta + 2$ is sharp, as follows from the Δ-star $K_{1,\Delta}$. Recently, it is shown in [14] that if $9 \leq \Delta \leq 11$, then $\chi_{vef}(G) \leq \Delta + 2$, and in [16] that if $10 \leq \Delta \leq 11$, then $\mathrm{ch}_{vef}(G) \leq \Delta + 2$. Other related results on entirely coloring plane graphs can be found in [4,6,7,12,18].

In this paper, we prove the following result, which is an improvement to the result in [2,16]:

Theorem 1.1. *If G is a plane graph with $\Delta = 9$, then $\mathrm{ch}_{vef}(G) \leq 11$.*

The organization of this paper is as follows. In Section 2, we define some notation and basic definitions used in the subsequent sections. In Section 3, we establish some structural lemmas, which play key roles in the proof of the main result. The proof of Theorem 1.1 is put in Section 4.

2. Concepts and notation

Let G be a plane graph with $\delta \geq 2$. For $f \in F$, if $b(f) = u_1u_2 \cdots u_nu_1$ is the boundary walk of f, then we simply write $f = [u_1u_2 \cdots u_n]$. Repeated occurrences of a vertex are allowed. The degree of a face is the number of edge-steps in its boundary walk. Note that each cut-edge is counted twice. For $x \in V \cup F$, let $d_G(x)$ (or $d(x)$, if no ambiguity) denote the degree of x in G. A vertex of degree k (at most k, at least k, respectively) is called a k-vertex (k^--vertex, k^+-vertex, respectively). Similarly, we can define k-face, k^--face and k^+-face. When v is a k-vertex, we say that there are k faces incident to v. However, these faces are not required to be distinct, i.e., v may have repeated occurrences on the boundary walk of some of its incident faces. v is often said to be a *$(a_1, \ldots, a_k)$-vertex* if it is cyclically incident to k faces $f_1, \ldots, f_k$ with $d_G(f_i) = a_i$ for $1 \leq i \leq k$. For an element $x \in V \cup F$ and an integer $i \geq 1$, let $n_i(x)$ ($m_i(x)$, respectively) denote the number of i-vertices (i-faces, respectively) adjacent or incident to x.

A 4-vertex v of G is called *weak* if $m_3(v) \geq 2$. A 4-face f of G is called *weak* if $2n_2(f) + n_3(f) + n_{4^w}(f) + m_3(f) \geq 2$. A 5-vertex v of G is called *weak* if $m_3(v) + m_{4^w}(v) = 5$, where 4^w denote the weak 4-face or weak 4-vertex. A 5^+-face f of G is called *weak* if

$$\beta(f) = 2n_2(f) + n_3(f) + n_{4^w}(f) + m_3(f) + m_{4^w}(f) \geq 3d_G(f) - 10. \tag{2.1}$$

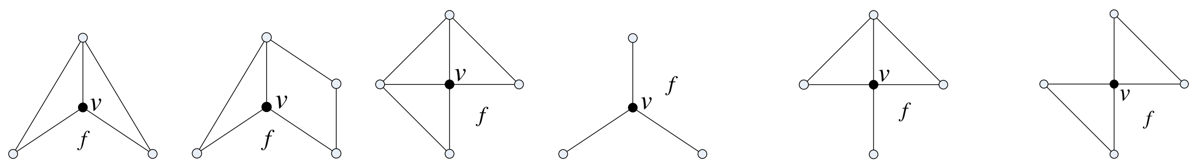
A cycle C of G is called *separating* if both its interior and exterior contain at least one vertex of G. Let $V^0(C)$ denote the set of vertices in G that lie in the interior of C. A 2-vertex is called *normal* if no separating 3-cycle passes through it, *worse* if it is incident to a 3-face, *bad* if it is incident to a 4-face, and *good* if no 4^--face is incident to it.

For an edge $e = xy \in E$, let $p(e)$ denote the total number of 3^--vertices, weak 4-vertices and weak 5-vertices in $\{x, y\}$. Let $q(e)$ denote the total number of 3-faces and weak faces incident to e. If $p(e) \geq 1, q(e) \geq 1$ and $d_G(x)+d_G(y)-p(e)-q(e) \leq 8$, then e is called a *light edge*.

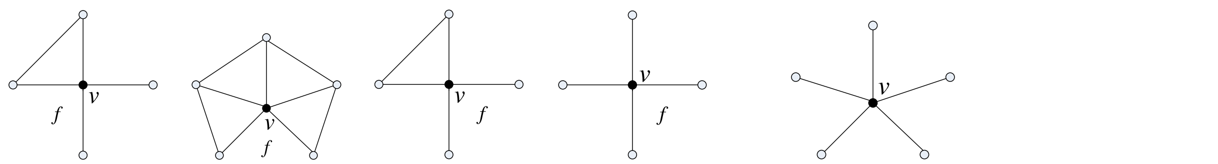
3. Structural lemmas

This section is devoted to establishing the following structural lemmas, which are used in the proof of the main result in next section.

Due to figures below, each of the configurations denotes the characteristics of the vertex v. The black vertex denotes a vertex of certain degree and the white vertices denote the vertices of uncertain degree. The triangle denotes a 3-face and the quadrangle denotes a 4-face. Meanwhile, other faces incident to v not indicated as a triangle or quadrangle are 4^+-faces, except the B_1-type and the C_3-type. In the following, we shall call a special vertex of some type. For example, the vertex v in the configuration B_1 is of B_1-type.



A_1 A_2 A_3 B_1 B_2 B_3



B_4 B_5 C_1 C_2 C_3: Here $m_3(v) \leq 3$