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On a packing problem of Alon and Yuster

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ABSTRACT

Two graphs G_1 and G_2, each on n vertices, *pack* if there exists a bijection f from $V(G_1)$ onto $V(G_2)$ such that $uv \in E(G_1)$ only if $f(u)f(v) \notin E(G_2)$. In 2014, Alon and Yuster proved that, for sufficiently large n, if $|E(G_1)| < n - \delta(G_2)$ and $\Delta(G_2) \leq \sqrt{n}/200$, then G_1 and G_2 pack. In this paper, we characterize the pairs of graphs for which the theorem of Alon and Yuster is sharp. We also prove the stronger result that for sufficiently large n, if $|E(G_1)| \leq n$, $\Delta(G_2) \leq \sqrt{n}/60$, and $\Delta(G_1) + \delta(G_2) \leq n - 1$, then G_1 and G_2 pack whenever there is a vertex $v_1 \in V(G_1)$ such that $d(v_1) = \Delta(G_1)$ and $\alpha(G_1 - N[v_1]) \geq \delta(G_2)$.

1. Introduction

Throughout this paper, the maximum degree and minimum degree of a vertex in a graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The size of a largest independent set in G is denoted by $\alpha(G)$.

Two graphs G_1 and G_2 with $|V(G_2)| = |V(G_1)|$ *pack* if there is a bijection $f : V(G_1) \to V(G_2)$ such that if $uv \in E(G_1)$, then $f(u)f(v) \notin E(G_2)$. In other words, graphs G_1 and G_2 pack if G_1 is a subgraph of the complement of G_2. Important results on graph packing were obtained in 1978 by Bollobás and Eldridge [2] and by Sauer and Spencer [6]. In particular, they proved that if two n-vertex graphs together contain at most $\frac{3}{2}n - 2$ edges, they are guaranteed to pack.

Theorem 1.1 ([2,6]). *Let G_1 and G_2 be two n-vertex graphs. If*

$$|E(G_1)| + |E(G_2)| \leq \frac{3}{2}n - 2, \tag{1}$$

then G_1 and G_2 pack.

Restriction (1) cannot be relaxed in view of the pair $\{G_1, G_2\}$ where G_1 is an n-vertex star and G_2 has no isolated vertices. Furthermore, Bollobás and Eldridge showed that if neither graph contains a star on n vertices, then (1) can be relaxed significantly.

Theorem 1.2 ([2]). *Let G_1 and G_2 be two n-vertex graphs. If $\Delta(G_1), \Delta(G_2) \leq n - 2$ and $|E(G_1)| + |E(G_2)| \leq 2n - 3$, then either G_1 and G_2 pack, or $\{G_1, G_2\}$ is one of the following 7 pairs: $\{2K_2, K_1 \cup K_3\}$, $\{\overline{K_2} \cup K_3, K_2 \cup K_3\}$, $\{3K_2, \overline{K_2} \cup K_4\}$, $\{\overline{K_3} \cup K_3, 2K_3\}$, $\{2K_2 \cup K_3, \overline{K_3} \cup K_4\}$, $\{\overline{K_4} \cup K_4, K_2 \cup 2K_3\}$, $\{\overline{K_5} \cup K_4, 3K_3\}$.*

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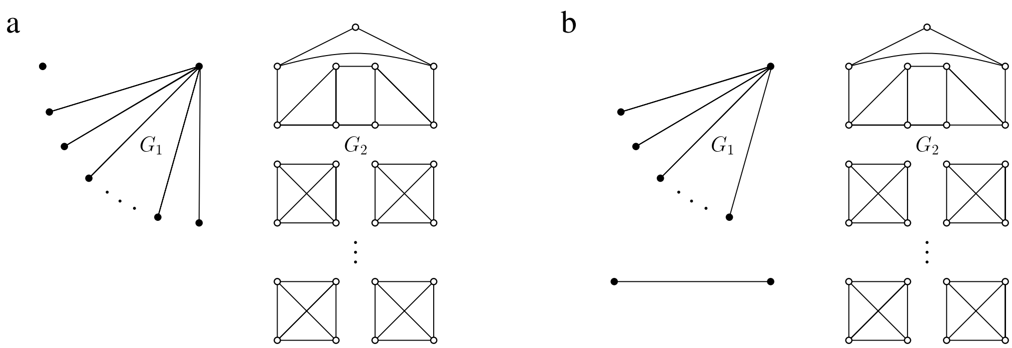

Fig. 1. Sharpness examples for Theorem 1.4 [1].

The restriction $2n-3$ in Theorem 1.2 is again sharp, since the cycle C_n does not pack with $K_{1,n-2}\cup K_1$ and, together, they have $2n-2$ edges. In a sense, Theorems 1.1 and 1.2 describe global properties of the graphs, since there are no restrictions on how the edges are arranged in the graph. On the other hand, the following result of Sauer and Spencer shows that two graphs even with many more edges will pack if their maximum degrees are not too large.

Theorem 1.3 (*[6]*). *Let G_1 and G_2 be two n-vertex graphs. If $\Delta(G_1)\Delta(G_2) < \frac{n}{2}$, then G_1 and G_2 pack.*

Recently, Alon and Yuster [1] considered packing a graph with few edges with a graph of bounded maximum degree.

Theorem 1.4 (*[1]*). *For all n sufficiently large, let G_1 and G_2 be n-vertex graphs such that $|E(G_1)| \le n-\delta(G_2)-1$ and $\Delta(G_2) \le \sqrt{n}/200$. Then G_1 and G_2 pack.*

Alon and Yuster phrased their theorem in the language of Turán numbers. The *Turán number* $\mathrm{ex}(n, G)$ of a graph G is the maximum number of edges in an n-vertex graph that does not contain a subgraph isomorphic to G. A result of Ore [5] from 1961 shows that $\mathrm{ex}(n, C_n) = \binom{n-1}{2} + 1$ and that for $n \ge 5$ the only graph with n vertices and $\binom{n-1}{2} + 1$ edges that does not contain a C_n is K_n minus a star with $n-2$ edges [5]. In this language, Theorem 1.4 is the following stronger version of Ore's result.

Theorem 1.5 (*[1]*). *For all n sufficiently large, if G is a graph of order n with no isolated vertices and $\Delta(G) \le \sqrt{n}/200$, then $\mathrm{ex}(n, G) = \binom{n-1}{2} + \delta(G) - 1$.*

Theorem 1.4 has the additional property that, unlike Ore's result, there are different sharpness examples. In particular, the following two examples are provided in [1], though we rephrase them in the language of graph packing. First, let G_1 be a star with $n-2$ edges and an additional vertex, that is $G_1 = K_{1,n-2}\cup K_1$. Let G_2 be a graph on n vertices in which all vertices but one have degree 3, the last vertex has degree 2 and the neighbors of this vertex are adjacent. Then G_1 has $n-\delta(G_2)$ edges, but the two graphs do not pack (Fig. 1(a)). Alternatively, if G_1 is the disjoint union of a star with $n-3$ vertices and an edge and G_2 remains unchanged, then G_1 and G_2 still do not pack (Fig. 1(b)).

In Fig. 1(a), $\Delta(G_1) + \delta(G_2) \ge n$, so G_1 and G_2 cannot pack since there is no suitable vertex in G_2 to which we might map the vertex of maximum degree in G_1. In Fig. 1(b), $\Delta(G_1) + \delta(G_2) = n-1$, so a potential packing could (and must) map the vertex of maximum degree in G_1 to the vertex of degree 2 in G_2. However, such an attempt will eventually fail to be a packing because no set of vertices could be mapped to the neighborhood of the degree 2 vertex.

With this observation, we can obtain a larger set of sharpness examples for Theorem 1.4. For example, fix constants n and d with n much larger than d. Let G_2 be a d-regular graph on n vertices consisting of a disjoint union of cliques. Let G_1 be the disjoint union of $d-1$ edges, together with a star containing $n-2(d-1)-1$ edges (Fig. 2(a), here $d = 6$). In fact, as long as there is no independent set of size d among the vertices in G_1 not in the star, we can create still more examples, e.g. Fig. 2(b).

The main result of this paper shows that if there is such an independent set of size $\delta(G_2)$, then G_1 and G_2 will pack even if G_1 contains as many as n edges.

Theorem 1.6. *For n sufficiently large ($n \ge 10^9$), let G_1 and G_2 be graphs of order n such that $\Delta(G_2) \le \sqrt{n}/60$, $|E(G_1)| \le n$, and $\Delta(G_1) + \delta(G_2) \le n-1$. If there is a vertex $v_1 \in V(G_1)$ such that*

$$d(v_1) = \Delta(G_1) \quad \text{and} \quad \alpha(G_1 - N[v_1]) \ge \delta(G_2), \tag{2}$$

then G_1 and G_2 pack.

Our theorem shows that if we are able to appropriately place the vertex of maximum degree in the sparse graph, then the remainder of the graph can also be placed. In fact, Theorem 1.6 is a generalization of Theorem 1.4. Indeed, if $|E(G_1)| \le n-\delta(G_2)-1$, then $\Delta(G_1) + \delta(G_2) \le n-1$. Also, if $v_1 \in V(G_1)$ with $d(v_1) = \Delta(G_1)$, then $G - N[v_1]$ contains