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# A rainbow Ramsey analogue of Rado's theorem

J.A. De Loera [a], R.N. La Haye [a,*], A. Montejano [b], D. Oliveros [c], E. Roldán-Pensado [c]

[a] Department of Mathematics, UC Davis, United States
[b] Facultad de Ciencias, UNAM campus Juriquilla, Mexico
[c] Instituto de Matemáticas, UNAM campus Juriquilla, Mexico

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ABSTRACT

We present a Rainbow Ramsey version of the well-known Ramsey-type theorem of Richard Rado. We use new techniques from the Geometry of Numbers. We also disprove two conjectures proposed in the literature.



## 1. Introduction

Given a set $X$, a *k-coloring* of $X$ is a surjective mapping $c : X \to \{1, 2, \ldots, k\}$, or equivalently a partition $X = C_1 \cup C_2 \ldots \cup C_k$ into $k$ nonempty parts called *color classes*. A subset $Y \subseteq X$ is called *monochromatic* under $c$ if it is contained in a single color class. On the other hand, $Y$ is called *rainbow* if the coloring assigns pairwise distinct colors to its elements. Given a coloring of a subset of the integers, we say that a vector in $X^k$, where $X$ is the subset of colored integers, is *rainbow* if each of its entries is colored differently.

*Arithmetic Ramsey Theory* concerns the study of the existence of monochromatic structures, like arithmetic progressions or solutions of linear equations, in every coloring of subsets of the integer numbers. The classical results in this area include Schur's Theorem: For every $k$, if $n$ is sufficiently large, every $k$-coloring of the starting segment of integers $[n] = \{1, 2, \ldots, n\}$ contains a monochromatic solution to the equation $x + y = z$. Another result is Van der Waerden's Theorem which states that for every pair of integers $t$ and $k$, when $n$ is sufficiently large, every $k$-coloring of $[n]$ contains a monochromatic $t$-term arithmetic progression. One of the most important examples is the famous 1933 theorem of Richard Rado: Given a rational matrix $A$, consider the homogeneous system of linear equations $Ax = 0$. This system or the matrix is called *k-regular* if, for every $k$-coloring of the natural numbers, the system has a monochromatic solution. A matrix is *regular* if it is $k$-regular for all $k$. Rado's Theorem characterizes precisely those matrices that are regular. The characterization depends on simple additive conditions satisfied by the columns of $A$ (which can be found in [12,9]). In fact, Rado's Theorem is a common generalization of both Schur's and Van der Waerden's Theorems.

* Corresponding author.
*E-mail addresses:* deloera@math.ucdavis.edu (J.A. De Loera), rlahaye@math.ucdavis.edu (R.N. La Haye), amandamontejano@ciencias.unam.mx (A. Montejano), dolivero@matem.unam.mx (D. Oliveros), e.roldan@im.unam.mx (E. Roldán-Pensado).

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In contrast to Ramsey Theory, *Rainbow Ramsey Theory* refers to the study of the existence of *rainbow* structures in colored combinatorial universes under some density conditions on the coloring. Arithmetic versions of this theory have been recently studied by several authors concerning colorings of integer intervals or cyclic groups, showing the existence of rainbow arithmetic progressions or rainbow solutions to linear equations under some density conditions on the color classes [5,8,10,11,7]. As pointed out in papers [5,8], one natural research direction is to generalize the known monochromatic results to the case of rainbow solutions of systems of linear equations. In particular the authors of [5] mention that it would be very exciting to provide a complete rainbow analogue of Rado's theorem. The key purpose of this paper is to provide such a theorem. As a consequence we disprove two conjectures formulated in [8,5] (Conjectures 1.2 and 1.3). Our techniques combine established combinatorial tools with ideas from convex geometry, particularly Ehrhart's Theory of lattice point counting [2,3].

**Definition 1.1.** A matrix $A$ with rational entries is *rainbow partition $k$-regular* if for all $n$ and for every equinumerous $k$-coloring of $[kn]$ (i.e. $k$-colorings in which all color classes have size $n$), there exists a rainbow vector in $\ker(A)$. The smallest $k$ such that $A$ is rainbow partition $k$-regular, if it exists, is called the *rainbow number of $A$* and is denoted by $r(A)$. A matrix $A$ is *rainbow regular* if it is rainbow partition $k$-regular for all sufficiently large $k$.

For instance, it is known that both matrices

$$A_1 = \begin{pmatrix} 1 & -2 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 1 & -1 & -1 \end{pmatrix},$$

corresponding to 3-term arithmetic progressions and solutions to the Sidon equation, are rainbow regular matrices with rainbow numbers $r(A_1) = 3$ and $r(A_2) = 4$ respectively (see [5,7]).

The authors of [5,8] claimed that every $1 \times n$ matrix with nonzero rational entries is rainbow regular if and only if some of the entries have different signs. That is correct for $n \geq 3$, but incorrect for $n = 2$. This subtle difference is crucial for finding the main theorem. Lemma 2.1 shows that all rational nonzero $1 \times 2$ matrices are not rainbow regular and is used to prove Theorem 1.5.

The papers [5,8] contain two different conjectures on the classification of rainbow regular matrices. However, both conjectures as originally stated have the trivial counterexample $\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$. To avoid this, we state them here in slightly modified versions.

**Conjecture 1.2** (*Jungić, Nešetřil, Radoičić [8]*)**.** *A matrix $A$ with integer entries is rainbow regular if and only if there exist two linearly independent vectors with distinct positive integer entries in* $\ker(A)$.

**Conjecture 1.3** (*Fox, Mahdian, Radoičić [5]*)**.** *A matrix $A$ with integer entries is rainbow regular if and only if the rows of $A$ are linearly independent and* $\ker(A)$ *contains a vector with distinct positive integer entries.*

After finding counterexamples to the above conjectures, we were able to obtain a Rado-style classification theorem of rainbow regular matrices. Moreover, the definition of rainbow regularity is stronger than it seems. We show that if $A$ is rainbow regular, it satisfies a stronger version of rainbow regularity where the equinumerous condition is relaxed.

**Definition 1.4.** A matrix $A$ with rational entries is *robustly rainbow regular* if there exists some constant $C$ such that for every $\varepsilon > 0$, positive integer $N$, and large enough integer $k$, the following holds: For every $k$-coloring of $[N]$ in which each color class contains at most $(C - \varepsilon)\frac{N}{\sqrt{k}}$ elements, there is a rainbow vector in $\ker(A)$.

Note that (robust) rainbow regularity is actually a property of the kernel rather than the matrix.

In our main theorem below, we classify rainbow regularity in terms of both the matrix $A$ and its kernel.

**Theorem 1.5.** *Let $A$ be a $m \times d$ rational matrix. The following conditions are equivalent.*

(i) *$A$ is rainbow regular.*
(ii) *$A$ is robustly rainbow regular.*
(iii) *There exists at least one vector in* $\ker(A)$ *with positive integer entries, and every submatrix of $A$ obtained by deleting two columns has the same rank as $A$.*
(iv) *There exists at least one vector in* $\ker(A)$ *with positive integer entries, and for every pair of distinct indices $(i, j)$, there exists a pair of vectors $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$ in* $\ker(A)$ *such that $x_i y_j \neq x_j y_i$.*

From Theorem 1.5 the reader can easily see that the matrix $\begin{pmatrix} a & -b & 0 \end{pmatrix}$, with $a, b$ positive integers gives a counterexample to Conjectures 1.2 and 1.3. However, not all counterexamples are this simple. For instance, the matrix

$$\begin{pmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 1 & -1 \end{pmatrix}$$

has a kernel generated by $(1, 2, 3, 4, 5)$ and $(1, 2, 4, 5, 6)$. By the existence of these two linearly independent vectors in $\ker(A)$, Conjectures 1.2 and 1.3 would imply that the matrix is rainbow regular. This is not the case because the submatrix of $A$ obtained by deleting the last two columns has the same rank as $A$, three.

In the next section we give the proof of Theorem 1.5 and we prove the following corollary about $k$-colorings of $\mathbb{N}$ instead of $[N]$.