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Note

Facial list colourings of plane graphs

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A B S T R A C T

Let $G = (V, E, F)$ be a connected plane graph, with vertex set V, edge set E, and face set F. For $X \in \{V, E, F, V \cup E, V \cup F, E \cup F, V \cup E \cup F\}$, two distinct elements of X are facially adjacent in G if they are incident elements, adjacent vertices, adjacent faces, or facially adjacent edges (edges that are consecutive on the boundary walk of a face of G). A list k-colouring is facial with respect to X if there is a list k-colouring of elements of X such that facially adjacent elements of X receive different colours. We prove that every plane graph $G = (V, E, F)$ has a facial list 4-colouring with respect to $X = E$, a facial list 6-colouring with respect to $X \in \{V \cup E, E \cup F\}$, and a facial list 8-colouring with respect to $X = V \cup E \cup F$. For plane triangulations, each of these results is improved by one and it is tight. These results complete the theorem of Thomassen that every plane graph has a (facial) list 5-colouring with respect to $X \in \{V, F\}$ and the theorem of Wang and Lih that every simple plane graph has a (facial) list 7-colouring with respect to $X = V \cup F$.

1. Introduction

All graphs considered in this paper are connected and may have loops or multiple edges. We use standard graph theory terminology according to Bondy and Murty [2]. We recall some important notions.

A *plane* graph is a particular drawing of a planar graph on the Euclidean plane. Suppose that $G = (V, E, F)$ is a connected plane graph with vertex set V, edge set E, and face set F. Faces of G are open 2-cells. The *boundary* of a face α is the boundary in the usual topological sense. It is a collection of all edges and vertices contained in the closure of the face α. Two vertices (two edges or two faces) are *adjacent* if they are joined by an edge (have a common end vertex or their boundaries have a common edge). A vertex and an edge are *incident* if the vertex is an end vertex of the edge. A vertex (an edge) and a face are *incident* if the vertex (the edge) lies on the boundary of the face.

A closed walk $W = v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k$ in G is a *boundary walk* corresponding to a face α if all vertices and edges of W are incident with α, and for all $i \in \{1, \ldots, k\}$, e_i follows e_{i-1} (indices modulo k) in the clockwise order of edges around v_i. In a connected plane graph, for every face α there exists a boundary walk containing all vertices and edges incident with α. This boundary walk is unique up to the choice of v_0 and e_0.

Two distinct edges are *facially adjacent* in G if they are consecutive edges on a boundary walk of a face of G.

A *proper k-entire-colouring* of a plane graph G is a mapping $\varphi : V \cup E \cup F \to \{1, \ldots, k\}$ such that any two distinct adjacent or incident elements in the set $V \cup E \cup F$ receive distinct colours. The *proper entire chromatic number* $\chi_{vef}(G)$ of a plane graph G is the smallest integer k such that G has a proper k-entire-colouring.

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Two distinct elements x and y of a graph are *facially adjacent* if they are incident elements, adjacent vertices, adjacent faces, or facially adjacent edges. A *facial k-entire-colouring* of a plane graph G is a mapping $\varphi : V \cup E \cup F \to \{1, \dots, k\}$ such that any two facially adjacent elements receive distinct colours. The *facial entire chromatic number* $\overline{\chi}_{vef}(G)$ of a plane graph is the smallest integer k such that G has a facial k-entire-colouring.

A mapping L is an *assignment* for the plane graph G if it assigns a list $L(x)$ of possible colours to each element x of $V \cup E \cup F$. If G has a proper (facial) entire colouring φ such that $\varphi(x) \in L(x)$ for all elements x, then we say that G is *properly* (*facially*) *L-entire-colourable* and φ is a *proper* (*facial*) *L-entire-colouring* of G. We say that G is *properly* (*facially*) *k-entire-choosable* if it is properly (facially) L-entire-colourable for any assignment L satisfying $|L(x)| = k$ for each element x of $V \cup E \cup F$. The *proper* (*facial*) *entire choice number* $ch_{vef}(G)$ ($\overline{ch}_{vef}(G)$) is the smallest k such that G is properly (facially) k-entire-choosable.

If we colour V, E, F, $V \cup E$, $E \cup F$, and $V \cup F$ (instead of $V \cup E \cup F$), then the corresponding colourings are called *vertex colouring, edge colouring, face colouring, total colouring, edge–face colouring, and coupled colouring*, of G, respectively. Facial and list versions for these colourings can be defined similarly.

All these (not necessarily facial) colourings of plane graphs have been studied extensively. For details concerning proper colourings see a recent survey by Borodin [11] and for facial colourings see a recent paper [14]. There are many deep results and also many challenging open problems.

This paper is devoted to study facial list colourings of plane graphs. There is a rich literature concerning proper list colourings of elements of plane graphs. Thomassen [20] proved that if G is planar, then $ch_v(G) = \overline{ch}_v(G) \le 5$, in other words, every planar graph is properly 5-vertex-choosable, whereas Voigt [22] presented an example of a non-4-vertex-choosable planar graph. Borodin proved in [6] that any plane graph G of maximum degree Δ is properly $(\Delta + 1)$-edge-choosable for $\Delta \ge 9$ and properly Δ-edge-choosable for $\Delta \ge 14$. The bound 14 was then lowered to 12 by Borodin, Kostochka, and Woodall [12]. Several recent results for $ch_e(G) \le \Delta + 1$ if $5 \le \Delta \le 8$ have been published for sparse graphs (often defined in terms of girth or collections of forbidden cycles), see [11,12,15,16,19,26,27].

Borodin proved in [4,5] that $ch_{ve}(G) \le \Delta + 2$ for $\Delta \ge 11$ and $\Delta \ge 9$, respectively, and improved the bound $\Delta + 2$ to $\Delta + 1$ if $\Delta \ge 16$. Borodin, Kostochka, and Woodall [12] proved that every plane graph is properly $(\Delta + 1)$-total-choosable if $\Delta \ge 12$.

The following conjecture of Woodall from 2009 (see [11]) is still open:

Conjecture 1. *If G is a simple plane graph with $\Delta(G) \ge 4$, then*

$$ch_{ve}(G) = \Delta + 1.$$

Wang and Lih [25] proved that any simple connected plane graph is properly 7-coupled-choosable and found out that there is a plane graph G with $\chi_{vf}(G) < ch_{vf}(G)$. Note that, by Borodin [3,9], $\chi_{vf}(G) \le 6$ for every plane graph G.

Borodin [8] proved that any simple plane graph G with $\Delta(G) = \Delta \ge 10$ is properly $(\Delta+1)$-edge–face-choosable. Wang and Lih [24] have proved that $ch_{ef}(G) \le \Delta + 3$ and constructed a plane graph H such that $\chi_{ef}(G) < ch_{ef}(G)$.

The list entire colouring was studied first by Borodin [10,7] who proved that any connected loopless and bridgeless plane graph G of maximum degree Δ is properly $(\Delta + 4)$-entire-choosable if $\Delta \ge 7$ and it is properly $(\Delta + 2)$-entire-choosable if $\Delta \ge 12$. The bound 12 is sharp. For $\Delta = 3$ the bound $ch_{vef}(G) \le \Delta + 4$ is tight.

In [14] it is proved that for any connected loopless and bridgeless plane graph G the following holds

$$\overline{\chi}_e(G) \le 4,\ \overline{\chi}_{ve}(G) \le 6,\ \overline{\chi}_{ef}(G) \le 6, \quad \text{and} \quad \overline{\chi}_{vef}(G) \le 8.$$

In this note we strengthen these results. Namely, we prove that any connected loopless and bridgeless plane graph is facially 4-edge-choosable, facially 6-total-choosable, and facially 8-entire-choosable. We also show that every plane triangulation is facially 3-edge-choosable, facially 5-total-choosable, and facially 7-entire-choosable, which are best possible.

The rest of the paper is organized as follows: In Section 2 we give some necessary notions and notation. Section 3 is devoted to facial list edge colourings. In Section 4 we study facial list total colourings of several families of plane graphs. Facial list entire colourings of plane graphs are investigated in Section 5.

2. Preliminaries

Given a plane graph $G = (V, E, F)$, one can define the *dual* $G^* = (V^*, E^*, F^*)$ of G as follows (see [2]): Corresponding to each face α of G there is a vertex α^* of G^*, and corresponding to each edge e of G there is an edge e^* of G^*. Two vertices α^* and β^* are joined by the edge e^* in G^* if and only if their corresponding faces α and β share the edge e in G. In the dual G^* of a plane graph G, the edges corresponding to those that lie on the boundary of a face α of G are just the edges incident with the corresponding vertex α^*. It is easy to see that the dual G^* of a plane graph G is itself a planar graph; in fact, there is a natural embedding of G^* on the plane. We place each vertex α^* in the corresponding face α of G, and then draw each edge e^* in such a way that it crosses the corresponding edge e of G exactly once (and crosses no other edge of G).

A graph is *k-degenerate* if it can be reduced to a single vertex by repeatedly deleting vertices of degree at most k (see [2]). It is easy to see that every k-degenerate graph G has the chromatic number $\chi(G) \le k+1$ and the choice number $ch(G)$ with $\chi(G) \le ch(G) \le k + 1$.