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Mixed orthogonal arrays, $(u, m, \boldsymbol{e}, s)$-nets, and $(u, \boldsymbol{e}, s)$-sequences

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ABSTRACT

We study the classes of $(u, m, \boldsymbol{e}, s)$-nets and $(u, \boldsymbol{e}, s)$-sequences, which are generalizations of (u, m, s)-nets and (u, s)-sequences, respectively. We show equivalence results that link the existence of $(u, m, \boldsymbol{e}, s)$-nets and so-called mixed (ordered) orthogonal arrays, thereby generalizing earlier results by Lawrence, and Mullen and Schmid. We use this combinatorial equivalence principle to obtain new results on the possible parameter configurations of $(u, m, \boldsymbol{e}, s)$-nets and $(u, \boldsymbol{e}, s)$-sequences, which generalize in particular a result of Martin and Stinson.

1. Introduction and basic definitions

The construction of point sets and sequences with good equidistribution properties is a classical problem in number theory and has important applications to quasi-Monte Carlo methods in numerical analysis (see the books of Dick and Pillichshammer [1], Leobacher and Pillichshammer [8], and Niederreiter [13]). The standard setting is that of the s-dimensional unit cube $[0, 1]^s$, for a given dimension $s \geq 1$, from which the points are taken. While the problem of constructing evenly distributed points in $[0, 1]^s$ is of number-theoretic origin, it also has a strong combinatorial flavor (see [1, Chapter 6] and [7, Chapter 15]).

Powerful methods for the construction of finite point sets with good equidistribution properties in $[0, 1]^s$ are based on the theory of nets (see again the references above as well as the original paper [12] and the recent handbook article [14]). This theory was recently extended by Tezuka [20] and studied in a slightly modified form by Hofer [3], Hofer and Niederreiter [4], Kritzer and Niederreiter [5], and Niederreiter and Yeo [17]. The underlying idea of these nets is to guarantee perfect equidistribution of the points for certain subintervals of the half-open unit cube $[0, 1)^s$. Concretely, for a dimension $s \geq 1$ and an integer $b \geq 2$, an interval $J \subseteq [0, 1)^s$ is called an *elementary interval in base b* if it is of the form

$$J = \prod_{i=1}^{s} [a_i b^{-d_i}, (a_i + 1) b^{-d_i}) \tag{1}$$

with integers $d_i \geq 0$ and $0 \leq a_i < b^{d_i}$ for $1 \leq i \leq s$. These intervals play a crucial role in the subsequent definition of a $(u, m, \boldsymbol{e}, s)$-net, which we state below. Here and in the following, we denote by $\mathbb{N}$ the set of positive integers and by λ_s the s-dimensional Lebesgue measure.

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Definition 1. Let $b \geq 2$, $s \geq 1$, and $0 \leq u \leq m$ be integers and let $\boldsymbol{e} = (e_1, \ldots, e_s) \in \mathbb{N}^s$. A point set $\mathcal{P}$ of b^m points in $[0, 1)^s$ is a *$(u, m, \boldsymbol{e}, s)$-net in base b* if every elementary interval $J \subseteq [0, 1)^s$ in base b of volume $\lambda_s(J) \geq b^{u-m}$ and of the form (1), with integers $d_i \geq 0$, $0 \leq a_i < b^{d_i}$, and $e_i | d_i$ for $1 \leq i \leq s$, contains exactly $b^m \lambda_s(J)$ points of $\mathcal{P}$.

Definition 1 is the definition of a $(u, m, \boldsymbol{e}, s)$-net in base b in the sense of [4]. Previously, Tezuka [20] introduced a slightly more general definition where the conditions on the number of points in the elementary intervals need to hold only for those elementary intervals J in base b with $\lambda_s(J) = b^{u-m}$. The narrower definition in [4] guarantees, as stated in that paper, that every $(u, m, \boldsymbol{e}, s)$-net in base b is also a $(v, m, \boldsymbol{e}, s)$-net in base b for every integer v with $u \leq v \leq m$. The latter property is very useful when working with such point sets (see again [4] for further details). Hence, whenever we speak of a $(u, m, \boldsymbol{e}, s)$-net here, we mean a $(u, m, \boldsymbol{e}, s)$-net in the narrower sense of Definition 1.

Note that the points of a $(u, m, \boldsymbol{e}, s)$-net tend to be very evenly distributed if u is small. But the choice of $e_1, \ldots, e_s \in \mathbb{N}$ also plays an important role since larger values of the e_i in general entail fewer restrictions in the defining property of the net.

For infinite sequences of points in $[0, 1]^s$ with good equidistribution properties, the corresponding concept is that of a $(u, \boldsymbol{e}, s)$-sequence. As usual, we write $[\boldsymbol{x}]_{b,m}$ for the coordinatewise m-digit truncation in base b of $\boldsymbol{x} \in [0, 1]^s$ (compare with [14, Remark 14.8.45] and [15, p. 194]).

Definition 2. Let $b \geq 2, s \geq 1$, and $u \geq 0$ be integers and let $\boldsymbol{e} \in \mathbb{N}^s$. A sequence $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ of points in $[0, 1]^s$ is a *$(u, \boldsymbol{e}, s)$-sequence in base b* if for all integers $g \geq 0$ and $m > u$, the points $[\boldsymbol{x}_n]_{b,m}$ with $gb^m < n \leq (g + 1)b^m$ form a $(u, m, \boldsymbol{e}, s)$-net in base b.

Again, the points of a $(u, \boldsymbol{e}, s)$-sequence are very evenly distributed if u is small, but also in this case the choice of $\boldsymbol{e}$ has an influence on the manner in which the points are spread over the elementary intervals in the unit cube.

If we choose $\boldsymbol{e} = (1, \ldots, 1) \in \mathbb{N}^s$ in Definitions 1 and 2, then these definitions coincide with those of a classical (u, m, s)-net and a classical (u, s)-sequence, respectively. The reasons why the more general $(u, m, \boldsymbol{e}, s)$-nets and $(u, \boldsymbol{e}, s)$-sequences were introduced have to do with their applications to quasi-Monte Carlo methods. Since this paper is devoted to the combinatorial aspects of $(u, m, \boldsymbol{e}, s)$-nets and $(u, \boldsymbol{e}, s)$-sequences, we do not elaborate on these reasons and we refer instead to [5, Section 1] and [20].

It was shown by Lawrence [6] and Mullen and Schmid [11] that classical (u, m, s)-nets are combinatorially equivalent to certain types of orthogonal arrays (see also [1, Section 6.2] for an exposition of this result). This equivalence has important implications for the theory of (u, m, s)-nets and (u, s)-sequences (see [1, Chapter 6] and [18]). The main result of the present paper generalizes this equivalence to $(u, m, \boldsymbol{e}, s)$-nets (see Theorem 5). The crucial step is to move from orthogonal arrays to mixed orthogonal arrays in the sense of [2, Chapter 9]. We recall the definition of a mixed orthogonal array $\mathrm{OA}\left(N, l_1^{k_1} \cdots l_v^{k_v}, t\right)$ from [2, Definition 9.1], where we change the notation from s_i to l_i since in our case s stands for a dimension. We write $R(b) = \{0, 1, \ldots, b-1\} \subset \mathbb{Z}$ for every integer $b \geq 2$.

Definition 3. Let $N \geq 1, l_1, \ldots, l_v \geq 2, k_1, \ldots, k_v \geq 1$, and $0 \leq t \leq k := k_1 + \cdots + k_v$ be integers. A *mixed orthogonal array* $\mathrm{OA}\left(N, l_1^{k_1} \cdots l_v^{k_v}, t\right)$ is an array of size $N \times k$ in which the first k_1 columns have symbols from $R(l_1)$, the next k_2 columns have symbols from $R(l_2)$, and so on, with the property that in any $N \times t$ subarray every possible t-tuple occurs an equal number of times as a row.

Remark 1. The parameter t of a mixed orthogonal array is called its *strength*. Definition 3 is vacuously satisfied for $t = 0$. As in [2, Definition 9.1], it is not required that $l_1, \ldots, l_v$ be distinct. If $l_1 = \cdots = l_v$, then Definition 3 reduces to that of an orthogonal array (see [2, Definition 1.1]).

Further results of this paper concern bounds on the parameters of $(u, m, \boldsymbol{e}, s)$-nets and $(u, \boldsymbol{e}, s)$-sequences for the case of greatest practical interest where $u = 0$ (see Theorems 1–4). Moreover, we show a necessary condition for the parameters of a mixed ordered orthogonal array (see Theorem 6) which generalizes [10, Lemma 3.1].

2. Necessary conditions for $(0, m, \boldsymbol{e}, s)$-nets

The parameter u of a $(u, m, \boldsymbol{e}, s)$-net is a nonnegative integer and its optimal value is $u = 0$. The following result imposes a combinatorial obstruction on the existence of $(0, m, \boldsymbol{e}, s)$-nets. If $\boldsymbol{e} = (e_1, \ldots, e_s) \in \mathbb{N}^s$, then we can assume without loss of generality that $e_1 \leq e_2 \leq \cdots \leq e_s$.

Theorem 1. *Let $\boldsymbol{e} = (e_1, \ldots, e_s) \in \mathbb{N}^s$ with $e_1 \leq e_2 \leq \cdots \leq e_s$. For $2 \leq t \leq s$ and $m \geq e_{s-t+1} + \cdots + e_{s-1} + e_s$, the existence of a $(0, m, \boldsymbol{e}, s)$-net in base b implies the existence of a mixed orthogonal array $\mathrm{OA}\left(b^m, l_1^1 \cdots l_s^1, t\right)$ with $l_i = b^{e_i}$ for $1 \leq i \leq s$.*

Proof. Let $\mathcal{P}$ be a $(0, m, \boldsymbol{e}, s)$-net in base b and let the points of $\mathcal{P}$ be

$$\boldsymbol{x}_n = (x_n^{(1)}, \ldots, x_n^{(s)}) \in [0, 1)^s \quad \text{for } n = 1, \ldots, b^m.$$