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Conjectured bounds for the sum of squares of positive eigenvalues of a graph

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ABSTRACT

A well known upper bound for the spectral radius of a graph, due to Hong, is that $\mu_1^2 \leq 2m - n + 1$ if $\delta \geq 1$. It is conjectured that for connected graphs $n - 1 \leq s^+ \leq 2m - n + 1$, where s^+ denotes the sum of the squares of the positive eigenvalues. The conjecture is proved for various classes of graphs, including bipartite, regular, complete q-partite, hyper-energetic, and barbell graphs. Various searches have found no counter-examples. The paper concludes with a brief discussion of the apparent difficulties of proving the conjecture in general.

1. Introduction

Let G be a simple and undirected graph with n vertices, m edges, chromatic number χ, minimum degree δ, maximum degree Δ and adjacency matrix A with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. The inertia of A is the ordered triple (ω, ν, γ), where ω, ν and γ are the numbers (counting multiplicities) of positive, negative and zero eigenvalues of A respectively. Let

$$s^+ = \sum_{i=1}^{\omega} \mu_i^2 \quad \text{and} \quad s^- = \sum_{i=n-\nu+1}^{n} \mu_i^2.$$

Note that $\sum_{i=1}^{n} \mu_i^2 = s^+ + s^- = tr(A^2) = 2m$ and $2m \geq 2(n-1)$ for connected graphs. Also let graph energy $E = \sum_{i=1}^{n} |\mu_i|$. Since $tr(A) = 0$,

$$\sum_{i=1}^{\omega} \mu_i = -\sum_{i=n-\nu+1}^{n} \mu_i = E/2.$$

Wocjan and Elphick [16] proved that $\chi \geq s^+/s^-$ and conjectured that $\chi \geq 1 + s^+/s^-$. This Conjecture was recently proven by Ando and Lin in [1]. It provides an example of replacing μ_1^2 with s^+, because Edwards and Elphick [6] proved that $\chi \geq 2m/(2m - \mu_1^2)$.

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In 1988 Hong [7] proved that for connected graphs:

$$\mu_1^2 \leq 2m - n + 1,$$

with equality only for K_n and Star graphs. Note that for K_n and Star graphs, $s^+ = \mu_1^2$. Hong [8] also noted that this bound holds for graphs with no isolated vertices.

This bound has been strengthened by several authors. For example, Nikiforov [11] proved that:

$$\mu_1 \leq \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(1+\delta)^2}{4}}$$

which is exact for various families of graphs, including regular graphs. It strengthens Hong's bound, as discussed in [11].

2. Conjecture

Conjecture 1. *Let G be a connected graph. Then*

$$\min(s^-, s^+) \geq n - 1.$$

Note that $s^- \geq n - 1$ implies that $s^+ \leq 2m - n + 1$ and vice versa.

Conjecture 2. *Let G be a graph with κ connected components. Then*

$$\min(s^-, s^+) \geq n - \kappa.$$

Proof. Let $G_1, \ldots, G_\kappa$ denote the components of G and let n_i denote the number of vertices in G_i. Then

$$s^-(G) = \sum s^-(G_i) \geq \sum (n_i - 1) = n - \kappa,$$

and similarly for $s^+(G)$. □

2.1. Comments

A graph is connected if and only if its adjacency matrix is irreducible. In the language of matrix algebra, this conjecture can therefore be expressed as $\min(s^-, s^+) \geq n - 1$ for binary, symmetric, irreducible matrices with zero trace.

Note that if L is the Laplacian of G, then $n - \kappa = rank(G) = rank(L) =$ number of positive eigenvalues of L.

We have searched the 10,000s of connected named graphs with 6 to 40 vertices in Wolfram Mathematica, and all connected graphs with up to 8 vertices, and found no counter-examples. A reviewer of this paper has also kindly checked all connected graphs with 9 and 10 vertices, and connected graphs with maximum degree four on 11 and 12 vertices and found no counter-example.

Note that for connected graphs, if $s^+ > s^-$ then $s^+ > m \geq n - 1$, and if $s^- > s^+$ then $s^- > m \geq n - 1$. Most, but not all graphs, have $s^+ \geq s^-$. So for any connected graph one half of the conjecture is true.

If we consider the set of connected graphs on n vertices, then it is notable that $s^- = n - 1$ for the graphs with the minimum number of edges (Trees) and the maximum number of edges (K_n).

Theorem 3. *Let G be any graph. Then $s^-(G) \leq n^2/4$.*

Proof. We use that $\mu_1 \geq 2m/n$ and assume that $s^- > n^2/4$, in which case:

$$2m = s^+ + s^- \geq \mu_1^2 + s^- \geq \frac{4m^2}{n^2} + s^- > \frac{4m^2}{n^2} + \frac{n^2}{4}.$$

This rearranges to:

$$0 > \left(\frac{2m}{n} - \frac{n}{2}\right)^2$$

which is a contradiction. □

Note that $s^- = \mu_n^2 = n^2/4$ for regular complete bipartite graphs. This bound can be compared with the following bound due to Constantine [4]:

$$\mu_n^2 \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \leq \frac{n^2}{4}.$$

2.2. An alternative formulation

The cyclomatic number, $c(G)$, is the minimum number of edges that need to be removed from a graph to make it acyclic. It is well known that $c = m - n + \kappa$, where κ is the number of components of a graph. We can therefore reformulate