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On a limit of the method of Tashkinov trees for edge-colouring

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ABSTRACT

The main technique used to edge-colour graphs requiring $\Delta + 2$ or more colours is the method of Tashkinov trees. We present a specific limit to this method, in terms of Kempe changes. We also provide a new Tashkinov tree extension.

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1. Introduction

Let G be a loopless graph. The chromatic index of G, denoted $\chi'(G)$, is the minimum number k of colours needed to k-edge-colour the graph—that is, to assign colours $\{1, 2, \ldots, k\}$ to the edges of G so that adjacent edges receive different colours. It is obvious that $\chi'(G) \geq \Delta$, where Δ denotes the maximum degree of G. If G is simple then Vizing's Theorem [14] tells us that $\chi'(G) \in \{\Delta, \Delta + 1\}$, however determining $\chi'(G)$ exactly is known to be NP-hard, even for simple cubic G [4]. A second lower bound for $\chi'(G)$ is given by the density parameter $\lceil \rho(G) \rceil$, where

$$\rho(G) = \max \left\{ \frac{2|E(G[S])|}{|S|-1} : S \subseteq V(G), |S| \text{ odd and } \geq 3 \right\}$$

(to see this notice that $\frac{|S|-1}{2}$ is the maximum size of a colour class in $G[S]$). Goldberg [3] and Seymour [11] conjectured that in general, $\chi'(G) \in \{\Delta, \Delta + 1, \lceil \rho(G) \rceil\}$ (or equivalently, $\chi'(G) \leq \max\{\lceil \rho(G) \rceil, \Delta + 1\}$). If this conjecture is true, then it would imply a polynomial-time algorithm for first checking whether or not $\chi'(G) > \Delta + 1$, and then, if the answer is yes, computing $\chi'(G)$ exactly—in contrast to the NP-hardness of edge-colouring. This implication is due to Edmonds [1]; see also [9] for an explanation. An asymptotic version of the Goldberg–Seymour conjecture is known (Kahn [5]), and the conjecture is known to be true for all graphs not containing a K_5^--minor (Marcotte [7]). Most other results towards the conjecture use the *method of Tashkinov trees* to establish an approximation—namely, to establish a result of the form $\chi'(G) \leq \max\{\lceil \rho \rceil, \Delta + s\}$ for some $s > 1$. For example, Scheide [10] proved such an approximation with $s = \frac{\Delta + 12}{14}$; see Stiebitz et al. [12] for a complete survey. In this paper we present a specific limit to the method of Tashkinov trees. We also provide a new Tashkinov tree extension.

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Suppose that you want to prove $\chi'(G) \leq \max\{\lceil\rho\rceil, \Delta+s\}$ for some $s \geq 1$, and to this end, you assume that $\chi'(G) > \Delta+s$. You succeed if you can find a subgraph H of G that is so dense that it requires more than $\chi'(G) - 1$ colours, that is, with $\lceil\rho(H)\rceil > \chi'(G) - 1$. This is because such a subgraph would imply that

$$\chi'(G) \geq \lceil\rho(G)\rceil \geq \lceil\rho(H)\rceil > \chi'(G) - 1,$$

and hence that $\chi'(G) = \lceil\rho(G)\rceil$, as desired. The method of Tashkinov trees, developed by Tashkinov [13] in 2000 as a common generalization of earlier work by Kierstead [6] and Vizing [14], suggests a candidate for H. In particular, starting with any partial edge-colouring of G, it describes how to perform (a polynomial number of) *Kempe changes* in order to find a candidate H. Given a partial k-edge-colouring of G, any pair of colours $a, b \in \{1, 2, \ldots, k\}$ induces a subgraph of G where each component is a path or an even cycle. A *Kempe change* is the action of switching a and b on any such component.

Let φ be a partial k-edge-colouring of G with $k \geq \Delta + 1$, which leaves at least one edge e_0 of G uncoloured. Define the set $W_{e_0}^{\varphi} \subseteq V(G)$ recursively as follows:

1. The two ends of e_0 are in $W_{e_0}^{\varphi}$.
2. Add a vertex v to $W_{e_0}^{\varphi}$ if there is a colour α such that (when considering φ): there is an edge e_v joining a vertex in $W_{e_0}^{\varphi}$ to v that is coloured α, and there is a vertex $u_v \in W_{e_0}^{\varphi}$ that is missing α (i.e. α is not used on any edge incident to u_v).

At any point during the recursive process, the current vertices in $W_{e_0}^{\varphi}$, along with e_0 and the edges that have been used in step 2, namely $\{e_v | v \in W_{e_0}^{\varphi}\}$, form a tree called a *$\varphi$-Tashkinov tree starting at e_0*. While there may be different maximal φ-Tashkinov trees starting at e_0 (owing to different possible choices for e_v), the vertex set of such a maximal tree, namely $W_{e_0}^{\varphi}$, is unique. *Tashkinov's Theorem* [13] says that either $W_{e_0}^{\varphi}$ is *φ-elementary* (no pair of vertices in $W_{e_0}^{\varphi}$ have a common missing colour), or that there is a sequence of Kempe changes that will modify φ so that both ends of e_0 have a common missing colour, and hence the colouring can be extended to e_0.

If $W_{e_0}^{\varphi}$ is φ-elementary, then each colour in φ either induces a near-perfect matching in $G[W_{e_0}^{\varphi}]$, or it is *defective*, that is, it occurs on more than one edge between $W_{e_0}^{\varphi}$ and $V(G) \setminus W_{e_0}^{\varphi}$. If there are no defective colours, then according to the definition of ρ,

$$\chi'(G) \geq \lceil\rho(G[W_{e_0}^{\varphi}])\rceil \geq \left\lceil \frac{2\left(k\left(\frac{|W_{e_0}^{\varphi}|-1}{2}\right)+1\right)}{|W_{e_0}^{\varphi}|-1} \right\rceil > k,$$

where the "+1" comes from the uncoloured edge e_0. In this situation, if $k < \chi'(G) - 1$, then one should add a new colour to φ and start the analysis again (i.e. consider again the resulting $W_{e_0}^{\varphi}$). On the other hand, if there are no defective colours and $k = \chi'(G) - 1$, then the above sequence of inequalities proves that $\chi'(G) = \lceil\rho(G)\rceil$, as desired.

We now know that in order to prove $\chi'(G) \leq \max\{\lceil\rho\rceil, \Delta + s\}$ via the method of Tashkinov trees, we "just" need to prove that, assuming $\chi'(G) > \Delta + s$, $W_{e_0}^{\varphi}$ has no defective colours, for e_0 some edge left uncoloured by φ, and φ some partial $(\chi'(G) - 1)$-edge-colouring with maximum domain (i.e. colouring the most edges among all partial $(\chi'(G) - 1)$-edge-colourings of G). When s is relatively large, this may not be too difficult: for any such colouring, the total number of colours in φ is $\chi'(G) - 1$, however each vertex in $W_{e_0}^{\varphi}$ has at least $\chi'(G) - 1 - \Delta \geq 1$ missing colours (all of which are distinct since the vertices of $W_{e_0}^{\varphi}$ are φ-elementary). If there is a defective colour, then it is in addition to the missing colours (by the maximality of our tree), so we get that

$$\chi'(G) - 1 \geq |W_{e_0}^{\varphi}|(\chi'(G) - 1 - \Delta) + 2 + 1,$$

where the "+2" is from the extra colours missing at the ends of e_0. Rearranging this equation, we get

$$\chi'(G) \leq \Delta + 1 + \frac{\Delta - 3}{|W_{e_0}^{\varphi}| - 1}.$$

Since $\chi'(G) > \Delta+s$, the larger $|W_{e_0}^{\varphi}|$ and/or s is, the tighter this bound will be. If we are able to argue that $|W_{e_0}^{\varphi}|$ is sufficiently large as to make this inequality invalid, then we get our desired approximation. Here, all of the work is in building a large Tashkinov tree.

As s gets smaller, it becomes more difficult to argue that a sufficiently large Tashkinov tree exists. In particular, it is easy to construct examples for $s = 1$ where $W_{e_0}^{\varphi}$ does have defective colours. Here, the natural approach is to do some Kempe changes to modify φ so that the new $W_{e_0}^{\varphi}$ is larger and does not have any defective colours, and this seems to work well in practice. However, is such a sequence of Kempe changes always possible? That is, can the basic method of Tashkinov trees described above possibly capture the density required to prove the Goldberg–Seymour Conjecture in general? In Section 2 we present an example which answers this question in the negative (Theorem 1). In fact the second author claimed such an example in her Ph.D. thesis [8], but a flaw was later found. Here we have a completely new example with no such failing.

The example of Section 2 does not mean the method of Tashkinov trees should be abandoned, but it does tell us that, without a technique beyond Kempe changes for modifying colourings, we cannot just rely on $W_{e_0}^{\varphi}$ to induce our dense subgraph H. Other authors, in particular Favrholt, Stiebitz and Toft [2], have already done work to define a set $W \supset W_{e_0}^{\varphi}$ that is φ-elementary and hence provides an improved candidate for H. We establish a new such result in Section 3.