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Interval edge-colorings of complete graphs

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ABSTRACT

An edge-coloring of a graph G with colors $1, 2, \ldots, t$ is an *interval t-coloring* if all colors are used, and the colors of edges incident to each vertex of G are distinct and form an interval of integers. A graph G is *interval colorable* if it has an interval t-coloring for some positive integer t. For an interval colorable graph G, $W(G)$ denotes the greatest value of t for which G has an interval t-coloring. It is known that the complete graph is interval colorable if and only if the number of its vertices is even. However, the exact value of $W(K_{2n})$ is known only for $n \leq 4$. The second author showed that if $n = p2^q$, where p is odd and q is nonnegative, then $W(K_{2n}) \geq 4n - 2 - p - q$. Later, he conjectured that if $n \in \mathbb{N}$, then $W(K_{2n}) = 4n - 2 - \lfloor \log_2 n \rfloor - \|n_2\|$, where $\|n_2\|$ is the number of 1's in the binary representation of n.

In this paper we introduce a new technique to construct interval colorings of complete graphs based on their 1-factorizations, which is used to disprove the conjecture, improve lower and upper bounds on $W(K_{2n})$ and determine its exact values for $n \leq 12$.

1. Introduction

All graphs in this paper are finite, undirected, have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G, respectively. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S, that is, $V(G[S]) = S$ and $E(G[S])$ consists of those edges of $E(G)$ for which both ends are in S. For a graph G, $\Delta(G)$ denotes the maximum degree of vertices in G. A graph G is *r-regular* if all its vertices have degree r. The set of edges M is called a *matching* if no two edges from M are adjacent. A vertex v is *covered* by the matching M if it is incident to one of the edges of M. A matching M is a *perfect matching* if it covers all the vertices of the graph G. The set of perfect matchings $\mathfrak{F} = \{F_1, F_2, \ldots, F_n\}$ is a *1-factorization* of G if every edge of G belongs to exactly one of the perfect matchings in $\mathfrak{F}$. The set of integers $\{a, a+1, \ldots, b\}$, $a \leq b$, is denoted by $[a, b]$. The terms, notations and concepts that we do not define can be found in [14].

A *proper edge-coloring* of graph G is a coloring of the edges of G such that no two adjacent edges receive the same color. The *chromatic index* $\chi'(G)$ of a graph G is the minimum number of colors used in a proper edge-coloring of G. If α is a proper edge-coloring of G and $v \in V(G)$, then the *spectrum* of a vertex v, denoted by $S(v, \alpha)$, is the set of colors of edges incident to v. By $\underline{S}(v, \alpha)$ and $\overline{S}(v, \alpha)$ we denote the smallest and largest colors of the spectrum, respectively. If α is a proper edge-coloring

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of G and H is a subgraph of G, then we can define a union and intersection of spectrums of the vertices of H:

$$S_{\cap}(H,\alpha) = \bigcap_{v\in V(H)} S(v,\alpha)$$
$$S_{\cup}(H,\alpha) = \bigcup_{v\in V(H)} S(v,\alpha).$$

A proper edge-coloring of a graph G with colors $1, 2, \dots, t$ is an *interval t-coloring* if all colors are used, and for any vertex v of G, the set $S(v,\alpha)$ is an interval of consecutive integers. A graph G is *interval colorable* if it has an interval t-coloring for some positive integer t. The set of interval colorable graphs is denoted by $\mathfrak{N}$. For a graph $G \in \mathfrak{N}$, the least and the greatest values of t for which G has an interval t-coloring are denoted by $w(G)$ and $W(G)$, respectively.

The concept of interval edge-coloring was introduced by Asratian and Kamalian [1]. In [1,2], they proved that if G is interval colorable, then $\chi'(G) = \Delta(G)$. For regular graphs the converse is also true. Moreover, if $G \in \mathfrak{N}$ is regular, then $w(G) = \Delta(G)$ and G has an interval t-coloring for every t, $w(G) \le t \le W(G)$. For a complete graph K_m, Vizing [13] proved that $\chi'(K_m) = m-1$ if m is even and $\chi'(K_m) = m$ if m is odd. These results imply that the complete graph is interval colorable if and only if the number of vertices is even. Moreover, $w(K_{2n}) = 2n-1$, for any $n \in \mathbb{N}$. On the other hand, the problem of determining the exact value of $W(K_{2n})$ is open since 1990.

In [6] Kamalian proved the following upper bound on $W(G)$:

Theorem 1. *If G is a connected graph with at least two vertices and $G \in \mathfrak{N}$, then $W(G) \le 2|V(G)| - 3$.*

This upper bound was improved by Giaro, Kubale, Malafiejski in [4]:

Theorem 2. *If G is a connected graph with at least three vertices and $G \in \mathfrak{N}$, then $W(G) \le 2|V(G)| - 4$.*

Improved upper bounds on $W(G)$ are known for several classes of graphs, including triangle-free graphs [1,2], planar graphs [3] and r-regular graphs with at least $2r+2$ vertices [7]. The exact value of the parameter $W(G)$ is known for even cycles, trees [5], complete bipartite graphs [5], Möbius ladders [10] and n-dimensional cubes [11,12]. This paper is focused on investigation of $W(K_{2n})$.

The first lower bound on $W(K_{2n})$ was obtained by Kamalian in [6]:

Theorem 3. *For any $n \in \mathbb{N}$, $W(K_{2n}) \ge 2n - 1 + \lfloor \log_2(2n-1) \rfloor$.*

This bound was improved by the second author in [11]:

Theorem 4. *For any $n \in \mathbb{N}$, $W(K_{2n}) \ge 3n - 2$.*

In the same paper he also proved the following statement:

Theorem 5. *For any $n \in \mathbb{N}$, $W(K_{4n}) \ge 4n - 1 + W(K_{2n})$.*

By combining these two results he obtained an even better lower bound on $W(K_{2n})$:

Theorem 6. *If $n = p2^q$, where p is odd, $q \in \mathbb{Z}_+$, then $W(K_{2n}) \ge 4n - 2 - p - q$.*

In that paper the second author also posed the following conjecture:

Conjecture 1. *If $n = p2^q$, where p is odd, $q \in \mathbb{Z}_+$, then $W(K_{2n}) = 4n - 2 - p - q$.*

He verified this conjecture for $n \le 4$, but the first author disproved it by constructing an interval 14-coloring of K_{10} in [8]. In "Cycles and Colorings 2012" workshop the second author presented another conjecture on $W(K_{2n})$:

Conjecture 2. *If $n \in \mathbb{N}$, then $W(K_{2n}) = 4n - 2 - \lfloor \log_2 n \rfloor - \|n_2\|$, where $\|n_2\|$ is the number of 1's in the binary representation of n.*

In Section 2 we show that the problem of constructing an interval coloring of a complete graph K_{2n} is equivalent to finding a special 1-factorization of the same graph. In Section 3 we use this equivalence to improve the lower bounds of Theorems 4 and 5, and disprove Conjecture 2. Section 4 improves the upper bound of Theorem 2 for complete graphs. In Section 5 we determine the exact values of $W(K_{2n})$ for $n \le 12$ and improve Theorem 6.

2. From interval colorings to 1-factorizations

Let the vertex set of a complete graph K_{2n} be $V(K_{2n}) = \{u_i, v_i \mid i = 1, 2, \dots, n\}$. For any fixed ordering of the vertices $\mathbf{v} = (u_1, v_1, u_2, v_2, \dots, u_n, v_n)$ we denote by $H_{\mathbf{v}}^{[i,j]}$, $i \le j$, the subgraph of K_{2n} induced by the vertices $u_i, v_i, u_{i+1}, v_{i+1}, \dots, u_j, v_j$.

Let $\mathfrak{F} = \{F_1, F_2, \dots, F_{2n-1}\}$ be a 1-factorization of K_{2n}. For every $F \in \mathfrak{F}$ we define its *left and right parts* with respect to the ordering of vertices $\mathbf{v}$:

$$l_{\mathbf{v}}^i(F) = F \cap E\left(H_{\mathbf{v}}^{[1,i]}\right)$$
$$r_{\mathbf{v}}^i(F) = F \cap E\left(H_{\mathbf{v}}^{[i+1,n]}\right).$$