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On fan–wheel and tree–wheel Ramsey numbers

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ABSTRACT

For graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer N such that, for any graph G of order N, G contains G_1 as a subgraph or the complement of G contains G_2 as a subgraph. Let T_n denote a tree of order n, W_n a wheel of order $n+1$ and F_n a fan of order $2n+1$. We establish Ramsey numbers for fans and trees versus wheels of even order, thereby extending several known results. In particular, we prove that $R(F_n, W_m) = 6n+1$ for odd $m \geq 3$ and $n \geq (5m+3)/4$, and that $R(T_n, W_m) = 3n-2$ for odd $m \geq 3$ and $n \geq m-2$, and T_n being a tree for which the Erdős–Sós Conjecture holds.

1. Introduction

For graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer N such that, for any graph G of order N, G contains G_1 as a subgraph or $\overline{G}$ contains G_2 as a subgraph, where $\overline{G}$ is the complement of G. Let $\chi(G)$ be the chromatic number of G and $s(G)$ the chromatic surplus of G, i.e., the minimum number of vertices in some color class under all proper vertex colorings with $\chi(G)$ colors. Let H be a connected graph of order p. Burr [6] established the following general lower bound for $R(H, G)$ when $p \geq s(G)$:

$$R(H, G) \geq (p-1)(\chi(G)-1) + s(G).$$

He also defined H to be G-good in case equality holds in this inequality.

A fan F_n consists of n triangles sharing exactly one common vertex. Thus, $|V(F_n)| = 2n+1$. A wheel W_m is the graph obtained from C_m and an additional vertex by joining it to every vertex of C_m. In this paper, our main aim is to consider when a fan F_n or a tree T_n is W_m-good for odd m. In order to do so, we first establish the following two auxiliary theorems.

Theorem 1. $R(nK_2, W_m) = \max\{2n + \lceil m/2 \rceil, n+m\}$.

Theorem 2. $R(nK_2, C_m) = \max\{2n + \lceil m/2 \rceil - 1, n+m-1\}$.

Remark. Even though Theorem 2 is not an immediate consequence of Theorem 1, the proofs of the two theorems use basically the same method. We therefore omit the proof of Theorem 2. The proof of Theorem 1 is postponed to Section 3.

For Ramsey numbers of fans versus wheels of even order, Surahmat et al. proved that F_n is W_3-good for $n \geq 3$, and obtained the following result.

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Theorem 3 (*Surahmat et al. [16]*)**.** *$R(F_n, W_3) = 6n + 1$ for $n \geq 3$.*

In our first main result we generalize the above result.

Theorem 4. *$R(F_n, W_m) = 6n + 1$ for odd $m \geq 3$ and $n \geq (5m + 3)/4$.*

The proof of Theorem 4 is postponed to Section 3.

For Ramsey numbers of trees versus odd cycles, Burr et al. confirmed that an arbitrary tree T_n on n vertices is C_m-good for odd $m \geq 3$ and $n \geq 756m^{10}$.

Theorem 5 (*Burr et al. [7]*)**.** *$R(T_n, C_m) = 2n - 1$ for odd $m \geq 3$ and $n \geq 756m^{10}$.*

As a tribute to Erdős and Sós, a tree T of order n is called an ES-tree if every graph $G = (V, E)$ with $|E| > |V|(n - 2)/2$ contains T as a subgraph. In 1963, Erdős and Sós conjectured that every tree is an ES-tree. Even though this conjecture is still open, it has been shown that many trees are indeed ES-trees. These results can be found in [10,15,9,17,13,11]. Two additional results were announced without being published. As mentioned in [14], Perles proved in 1990 that caterpillars are ES-trees, where a caterpillar is a tree in which a single path is incident to (or contains) every edge. Ajtai et al. [1] announced that all sufficiently large trees are ES-trees.

In the following, we first present a result on Ramsey numbers of ES-trees versus odd cycles. This turns out to be easy to prove, as is clear from the proof in Section 3. Using this result, we establish a theorem on Ramsey numbers of ES-trees versus wheels of even order. By this result we generalize (to some extent) results on Ramsey numbers of wheels versus special trees, including stars, paths and star-like trees. All proofs are postponed to Section 3.

Theorem 6. *$R(T_n, C_m) = 2n - 1$ for every ES-tree T_n, odd $m \geq 3$ and $n \geq m - 1$.*

Theorem 7. *$R(T_n, W_m) = 3n - 2$ for every ES-tree T_n, odd $m \geq 3$ and $n \geq m - 2$.*

From Theorem 7 a more general result can be obtained by induction.

Theorem 8. *$R(T_n, K_\ell + C_m) = (\ell + 2)(n - 1) + 1$ for every ES-tree T_n, odd $m \geq 3$, $\ell \geq 1$ and $n \geq m - 2$.*

We conjecture that the above three theorems hold for all trees. If so, it provides more evidence that the Erdős–Sós Conjecture is true.

Terminology will in general follow that used in [3]. In particular, the length of a longest and shortest cycle of G is denoted by $c(G)$ and $g(G)$, respectively. A graph G is weakly pancyclic if it contains cycles of every length between $g(G)$ and $c(G)$. For two vertex-disjoint graphs G_1 and G_2, $G_1 \cup G_2$ denotes the disjoint union, and the join $G_1 + G_2$ is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 to every vertex of G_2 by an edge. We use mG to denote m vertex-disjoint copies of G.

2. Lemmas

In order to prove our main theorems, we need the following results.

Lemma 1 (*Brandt [4]*)**.** *Every nonbipartite graph G of order n with $e(G) > (n - 1)^2/4 + 1$ is weakly pancyclic with $g(G) = 3$.*

Lemma 2 (*Brandt et al. [5]*)**.** *Every nonbipartite graph G of order n with $\delta(G) \geq (n + 2)/3$ is weakly pancyclic with $g(G) = 3$ or 4.*

Lemma 3 (*Dirac [8]*)**.** *Let G be a graph with $\delta(G) \geq 2$. Then $c(G) \geq \delta(G) + 1$. Moreover, if $\delta(G) \geq |V(G)|/2$, then G has a Hamilton cycle.*

Lemma 4 (*Erdős and Gallai [10]*)**.** *Let G be a graph of order n and $3 \leq c \leq n$. If $e(G) \geq 1/2((c - 1)(n - 1) + 1)$, then $c(G) \geq c$.*

Lemma 5 (*Erdős and Gallai [10]*)**.** *Let G be a graph of order n. If $e(G) > n(k - 2)/2$, then G contains a path on k vertices.*

Lemma 6 (*Hasmawati et al. [12]*)**.** *If m is odd and $3 \leq m \leq 2n - 1$, then $R(K_{1,n-1}, W_m) = 3n - 2$.*

Lemma 7 (*Baskoro et al. [2]*)**.** *For odd $n \geq 3$, let G be a graph of order n which is obtained from K_n by removing $\lfloor n/2 \rfloor$ independent edges. Then G contains all trees on n vertices.*

The following lemma is an established result that is straightforward to prove using a Greedy algorithm.

Lemma 8. *Let G be a graph with $\delta(G) \geq k$, and let $u \in V(G)$. Let T be a tree of order $k + 1$ with $v \in V(T)$. Then T can be embedded in G such that v is mapped to u.*