Discrete Mathematics 339 (2016) 2304–2312

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Finite families of forbidden subgraphs for rainbow connection in graphs

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ARTICLE INFO

Article history:
Received 28 January 2015
Received in revised form 19 February 2016
Accepted 23 February 2016
Available online 6 May 2016

Keywords:
Rainbow connection
Forbidden subgraphs
Edge-coloring

ABSTRACT

A connected edge-colored graph G is rainbow-connected if any two distinct vertices of G are connected by a path whose edges have pairwise distinct colors; the rainbow connection number $\mathrm{rc}(G)$ of G is the minimum number of colors such that G is rainbow-connected. We consider families $\mathcal{F}$ of connected graphs for which there is a constant $k_{\mathcal{F}}$ such that, for every connected $\mathcal{F}$-free graph G, $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$, where $\mathrm{diam}(G)$ is the diameter of G. In the paper, we finalize our previous considerations and give a complete answer for any finite family $\mathcal{F}$.

1. Introduction

We consider finite and simple graphs only, and for terminology and notation not defined here we refer to [3]. To avoid trivial cases, all graphs considered will be connected with at least one edge.

An edge-colored connected graph G is said to be *rainbow-connected* if each pair of distinct vertices of G is connected by a rainbow path, i.e., by a path whose edges have pairwise distinct colors. Note that the edge coloring need not be proper. The *rainbow connection number* of G, denoted by $\mathrm{rc}(G)$, is the minimum number of colors such that G is rainbow-connected.

The concept of rainbow connection was introduced by Chartrand et al. [7]. It is easy to observe that, for any graph G, $\mathrm{rc}(G) \leq |V(G)| - 1$, since we can color the edges of a given spanning tree of G with different colors, and the remaining edges with one of the already used colors. In [7], the exact values of $\mathrm{rc}(G)$ were determined for several graph classes. The rainbow connection number has been studied for further graph classes in [4,8,10,13] and for graphs with fixed minimum degree in [4,11,17]. The results are surveyed in [14] and in [15].

In [5,12], it was shown that it is NP-hard to determine the exact value of $\mathrm{rc}(G)$. In fact, it is already NP-complete to decide whether $\mathrm{rc}(G) = 2$, and it is also NP-complete to decide whether a given edge-colored graph (with an unbounded number of colors) is rainbow-connected [5]. More generally, it has been shown in [12] that for any fixed $k \geq 2$, it is NP-complete to decide whether $\mathrm{rc}(G) = k$.

For the rainbow connection numbers of graphs the following results are known (and obvious).

Proposition A. *Let G be a connected graph of order n. Then*

(i) $1 \leq \mathrm{rc}(G) \leq n - 1$,
(ii) $\mathrm{rc}(G) \geq \mathrm{diam}(G)$,

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http://dx.doi.org/10.1016/j.disc.2016.02.015
0012-365X/

(iii) $\mathrm{rc}(G) = 1$ *if and only if* G *is complete,*
(iv) $\mathrm{rc}(G) = n - 1$ *if and only if* G *is a tree.*

Note that the difference $\mathrm{rc}(G) - \mathrm{diam}(G)$ can be arbitrarily large. For $G \simeq K_{1,n-1}$ we have $\mathrm{rc}(K_{1,n-1}) - \mathrm{diam}(K_{1,n-1}) = (n-1) - 2 = n - 3$. Especially, each bridge requires a single color.

For connected bridgeless graphs, there is the following upper bound on $\mathrm{rc}(G)$, however, note that this bound is quadratic in terms of $\mathrm{rad}(G)$, and, since there is a constant c such that $c \cdot \mathrm{rad}(G) \geq \mathrm{diam}(G)$, also in $\mathrm{diam}(G)$.

Theorem B ([1]). *For every connected bridgeless graph G with radius r,*

$$\mathrm{rc}(G) \leq r(r+2).$$

Moreover, for every integer $r \geq 1$*, there exists a bridgeless graph G with radius r and* $\mathrm{rc}(G) = r(r+2)$.

Let $\mathcal{F}$ be a family of connected graphs. We say that a graph G is $\mathcal{F}$*-free* if G does not contain an induced subgraph isomorphic to a graph from $\mathcal{F}$. Specifically, for $\mathcal{F} = \{X\}$, we say that G is X*-free*, and for $\mathcal{F} = \{X_1, \dots, X_k\}$, we say that G is $(X_1, \dots, X_k)$*-free*. The members of $\mathcal{F}$ will be referred to in this context as *forbidden induced subgraphs*. If $\mathcal{F} = \{X_1, \dots, X_k\}$, we will also refer to the graphs $X_1, \dots, X_k$ as a *forbidden k-tuple*, and for $|\mathcal{F}| = 2$, 3 and 4, we will also speak about *forbidden pair*, *forbidden triple* and *forbidden quadruple*, respectively.

Graphs characterized in terms of forbidden induced subgraphs are known to have many interesting properties. Although, as we know from Theorem B, $\mathrm{rc}(G)$ can be quadratic in terms of $\mathrm{diam}(G)$ even in bridgeless graphs, it turns out that the upper bound on $\mathrm{rc}(G)$ in terms of $\mathrm{diam}(G)$ can be remarkably lowered under forbidden subgraph conditions.

In [9], the authors considered the question for which families $\mathcal{F}$ of connected graphs, a connected $\mathcal{F}$-free graph satisfies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$, where $k_{\mathcal{F}}$ is a constant (depending on $\mathcal{F}$), and gave a complete answer for $1 \leq |\mathcal{F}| \leq 2$ by the following two results (where N denotes the *net*, i.e. the graph obtained by attaching a pendant edge to each vertex of a triangle).

Theorem C ([9]). *Let X be a connected graph. Then there is a constant* k_X *such that every connected X-free graph G satisfies* $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_X$*, if and only if* $X = P_3$.

Theorem D ([9]). *Let* X, Y *be connected graphs,* $X, Y \neq P_3$*. Then there is a constant* k_{XY} *such that every connected* (X, Y)*-free graph G satisfies* $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{XY}$*, if and only if (up to a symmetry) either* $X = K_{1,r}$*,* $r \geq 4$ *and* $Y = P_4$*, or* $X = K_{1,3}$ *and Y is an induced subgraph of N.*

In this paper, we will consider an analogous question for a finite family $\mathcal{F}$ with $|\mathcal{F}| \geq 3$. Namely, we will consider the following question.

For which finite families $\mathcal{F} = \{X_1, X_2, \dots, X_k\}$ (where $k \geq 3$ is an integer) of connected graphs, there is a constant $k_{\mathcal{F}}$ such that a connected graph G being $\mathcal{F}$-free implies $\mathrm{rc}(G) \leq \mathrm{diam}(G) + k_{\mathcal{F}}$?

We give a complete characterization for $|\mathcal{F}| = 3$ in Theorem 1, for $|\mathcal{F}| = 4$ in Theorem 9, and for an arbitrary finite family $\mathcal{F}$ in Theorem 10.

2. Preliminary results

In this section we summarize some further notations and facts that will be needed for the proofs of our results.

If G is a graph and $A \subset V(G)$, then $G[A]$ denotes the subgraph of G induced by the vertex set A, and $G - A$ the graph $G[V(G) \setminus A]$. Specifically, for $x \in V(G)$, $G - x$ is the graph $G[V(G) \setminus \{x\}]$, and for $e \in E(G)$, $G - e$ is the graph obtained from G by deleting the edge e. An edge $e \in E(G)$ such that $G - e$ is disconnected is called a *bridge*, and a graph with no bridges is called a *bridgeless graph*. An edge such that one of its vertices has degree one is called a *pendant edge*. For $x, y \in V(G)$, a path in G from x to y will be referred to as an (x, y)*-path*, and, whenever necessary, it will be considered with orientation from x to y. For a subpath of a path P with origin u and terminus v (also referred to as a (u, v)*— arc of P*), we will use the notation uPv. Similarly, if C is a cycle with a fixed orientation, then uCv denotes the arc of C with origin u and terminus v, in the given orientation of C. If x is a vertex of a path or of a cycle (with a fixed orientation), then x^- and x^+ denote its predecessor and successor, respectively.

For graphs X and G, we write $X \subset G$ if X is a subgraph of G, $X \overset{\mathrm{IND}}{\subset} G$ if X is an induced subgraph of G, and $X \simeq G$ if X and G are isomorphic. We use $d_G(x)$ for the degree of a vertex x, and, for two vertices $x, y \in V(G)$, we denote by $\mathrm{dist}(x, y)$ the distance between x and y in G. The diameter and the radius of a graph G will be denoted by $\mathrm{diam}(G)$ and $\mathrm{rad}(G)$, respectively. A shortest path joining two vertices at distance $\mathrm{diam}(G)$ will be referred to as a *diameter path*. We use $\alpha(G)$ for the independence number of G, $\overline{G}$ for the complement of a graph G, $\delta(G)$ for the minimum degree of G, and $\overline{\delta}(G)$ for the average degree of G (i.e., $\overline{\delta}(G) = \frac{1}{|V(G)|} \sum_{x \in V(G)} d_G(x)$). Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers.

For a set $S \subset V(G)$ and $k \in \mathbb{N}$, the *neighborhood at distance k* of S is the set $N_G^k(S)$ of all vertices of G at distance k from S. In the special case when $k = 1$, we simply write $N_G(S)$ for $N_G^1(S)$, and if $|S| = 1$ with $x \in S$, we write $N_G(x)$ for $N_G(\{x\})$. For a set $M \subset V(G)$, we set $N_M(S) = N_G(S) \cap M$ and $N_M(x) = N_G(x) \cap M$, and for a subgraph $P \subset G$, we write $N_P(x)$ for